

Stable and Unstable Operations in Algebraic Cobordism

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Abstract

We describe additive (unstable) operations from a theory A^* obtained from Algebraic Cobordism Ω^* of M. Levine-F. Morel by change of coefficients to any oriented cohomology theory B^* . We prove that there is 1-to-1 correspondence between the set of operations, and the set of transformations: $A^n((\mathbb{P}^\infty)^{\times r}) \rightarrow B^m((\mathbb{P}^\infty)^{\times r})$ satisfying certain simple properties. This provides an effective tool of constructing such operations. As an application, we prove that (unstable) additive operations in Algebraic Cobordism are in 1-to-1 correspondence with the $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ -linear combinations of Landweber-Novikov operations which take integral values on the products of projective spaces. On our way we obtain that stable operations there are exactly \mathbb{L} -linear combinations of Landweber-Novikov operations. We also show that multiplicative operations $A^* \rightarrow B^*$ are in 1-to-1 correspondence with the morphisms of the respective formal group laws. We construct Integral (!) Adams Operations in Algebraic Cobordism, and all theories obtained from it by change of coefficients, giving classical Adams operations in the case of K_0 . Finally, we construct Symmetric Operations for all primes p (these operations in Ω^* , previously known only for $p = 2$, are more subtle than the Landweber-Novikov operations, and have applications to rationality questions - [19],[18],[20]), as well as the T. tom Dieck - style Steenrod operations in Algebraic Cobordism.

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1 Introduction

In the current article we study operations between Generalized Oriented Cohomology Theories. An interest in this subject arose from the prominent role which cohomological operations play in Topology, as well as from the already known successful applications of them in Algebro-Geometric context. Not mentioning the classical applications of Adams operations in Algebraic K-theory, it starts with the crucial use of Milnor's operations in motivic cohomology by V.Voevodsky in his proof of Milnor's Conjecture ([23] and [22]). In the process he had to construct the whole Steenrod algebra in motivic cohomology with \mathbb{Z}/l -coefficients. The restriction of this algebra to Chow groups modulo l was later produced by P.Brosnan ([3]) by an elementary construction using equivariant Chow groups. Meanwhile, I.Panin and A.Smirnov ([11],[12]) were studying multiplicative operations between arbitrary theories, and their relation to the orientation, resulting in such statements as the general Riemann-Roch Theorem. Finally, M.Levine and F.Morel produced the universal generalized oriented cohomology theory - the Algebraic Cobordism Ω^* ([8],[9]). The universality of Ω^* combined with the reorientation procedure of I.Panin-A.Smirnov (following D.Quillen [14]) permitted to produce the multiplicative operations $\Omega^* \rightarrow B^*$ easily and to classify them (in the "invertible" case). In particular, one gets that all such operations are specializations of the Total Landweber-Novikov operation $\Omega^* \rightarrow \Omega^*[b_1, b_2, \dots]$. And the resulting Landweber-Novikov algebra plays an important role in the study of Algebraic Cobordism, and other *free* theories of M.Levine-F.Morel. In all the mentioned cases, with the exception of Adams operations, the operations were (essentially) stable. As a rare example of unstable operations (in the algebro-geometric context), the, so-called, Symmetric operations (mod 2) were introduced in [17] and [19]. Originally constructed with the aim of producing maps between Chow groups of different quadratic Grassmannians (of the same quadratic form), these operations in Algebraic Cobordism of M.Levine-F.Morel were successfully applied to the question of rationality of algebraic cycles ([18],[20]), where they provide the only known method to deal with 2-torsion. These operations can be combined into a Total one which is a "formal half" of the "negative part" of the Total Steenrod operation (mod 2) in Algebraic Cobordism. The topological counterpart of it was apparently used by D.Quillen in [14]. Being more subtle than the Landweber-Novikov operations, the Symmetric ones (mod 2), in some sense, "plug the gap" between \mathbb{L} and $H_*(MU)$ left by the Hurewicz map, plug 2-adically. To have an integral variant of such statements one would need Symmetric operations for all primes. Unfortunately, the case $p = 2$ was produced by an explicit geometric construction (using $Hilb_2$), and it is unclear how to extend it for other primes. The desire to construct these operations was the main motivation behind the current article. In the end, it appeared that to produce Symmetric operation for $p > 2$ is about as "simple" as to produce all (unstable) additive operations in Algebraic Cobordism. But to do it, one has to develop some new tools. To start with, one has to understand better the theory itself - what is it which distinguishes it from other theories of "low quality"? What, I think, was underestimated previously, is that, in contrast to Topology, the general

object satisfying the Definition of the Generalized Oriented Cohomology Theory is not particularly good. And so, if one wants to obtain a result reminiscent of a topological one, one has to restrict attention to a very special class of theories. The picture which emerges suggests that there are "topological quality" theories parameterized by a formal group law $\mathbb{L} \rightarrow A$, and a non-negative integer n , while "inbetween" and "across" there is an ocean of "low-grade" theories. The property of Algebraic Cobordism which permits us to deal with the unstable operations successfully, is that this theory can be defined inductively on dimension. Namely, that $\Omega^*(X)$ can be described in terms of Ω^* of smaller-dimensional varieties (and some explicit relations). This leads to the notion of a *theory of rational type* (= *type 0*). Such theories appear to be the same as *free theories* of M.Levine-F.Morel, and are the best theories available there. In particular, all the "standard" theories, like, CH , K_0 , BP , higher Morava's K-theories $K(n)$ are of this sort. At this stage I should recall that there are two types of cohomology theories in Algebraic Geometry: the "large" ones $A^{j,i}$ - represented by some spectrum in \mathbb{A}^1 -homotopy theory, numbered by two indices, and "small" ones A^i , typically, represented by the "pure parts" $A^{2i,i}$ of large theories. The Algebraic Cobordism Ω^* of M.Levine-F.Morel belongs to the second type and, by the result of M.Levine ([7]), is represented by the pure part of MGL of V.Voevodsky. In this article, we work with "small" theories. The fact that Ω^* is a *theory of rational type* is non-trivial. Our proof uses the mentioned comparison result of M.Levine ([7]), but we hope to by-pass it in the later version of the text (which would make our treatment "internal" for Ω^*). Any theory A^* of rational type can be inductively described in terms of generators (of smaller dimension) and relations. We provide three alternative descriptions here: two in terms of push-forwards, and one in terms of pull-backs - see Subsections 4.1,4.2,4.3. After that it becomes possible to construct operations inductively on dimension. This enables us to show that not only each operation $A^n \rightarrow B^m$ is completely determined by its action on $(\mathbb{P}^\infty)^{\times r}$, for all r (Proposition 3.15), but also that if one is given the transformation: $A^n((\mathbb{P}^\infty)^{\times r}) \rightarrow B^m((\mathbb{P}^\infty)^{\times r})$, for all r , commuting with the action of the symmetric group, the partial diagonals, and the partial Segre embeddings, then it extends to the unique operation $A^n \rightarrow B^m$. This is our Main result Theorem 5.1. The "multiplicative" variant of it (Proposition 5.17) says that multiplicative operations correspond to transformations as above commuting also with the external products of projective spaces. These results permit to describe and construct operations effectively, as one only needs to define them on $(\mathbb{P}^\infty)^{\times r}$, which is a cellular space. As a first application, we describe all additive (unstable) operations in Algebraic Cobordism of M.Levine-F.Morel - see Theorem 6.1. These appears to be exactly those $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ -linear combinations (infinite, in general) of the Landweber-Novikov operations which take "integral" values on $\Omega^*((\mathbb{P}^\infty)^{\times r})$, for all r . The stable ones among them will be exactly \mathbb{L} -linear combinations - see Theorem 3.11, but this fact is much simpler, and does not require the above technique. In the case of multiplicative operations, we show (Theorem 6.8) that such operations $A^* \rightarrow B^*$ (where A^* is of rational type) are in 1-to-1 correspondence with the homomorphisms of the respective formal group laws. This reduces the classification of such multiplicative operations to algebra. In particular, we extend the result of I.Panin-A.Smirnov-M.Levine-F.Morel on multiplicative operations $\Omega^* \rightarrow B^*$ to the case where b_0 is not a zero divisor (compare with: " b_0 -invertible", as in the original Theorem). We show (in Theorem 6.9) that the reparametrization $\gamma = b_0x + b_1x^2 + \dots \in B[[x]]$ comes from a multiplicative operation as above if and only if the shifted formal group law F_B^γ has coefficients in B (has no denominators). As an immediate application of this we construct Integral Adams operations Ψ_k in Algebraic Cobordism and all other theories of rational type. Indeed, in this case, γ is a formal multiplication by k , so the shifting does not change F_B at all. These unstable multiplicative operations are A -linear, and are all obtained from the ones in Algebraic Cobordism by change of coefficients. In the case of K_0 we get the classical Adams operations. Similar considerations permit to construct the T.tom Dieck - style Steenrod operations in Algebraic Cobordism - Theorem 6.17 (an object more subtle than the Quillen's style Steenrod operations - see Subsection 6.4). Finally, using the Main Theorem 5.1 itself we construct Symmetric operations for all primes p - see Theorem 6.18. The last two results form a separate paper [21], not to overburden the given text. Aside

from the mentioned major results we present various smaller ones - see Sections 6 and 7. In particular, we show that all operations in Chow groups mod p are essentially stable (each extends to a unique stable operation), and consist of Steenrod operations only (Theorem 6.6). Finally, noting that topological application in Algebraic Geometry so far were restricted to simply pulling theories from Topology, thus creating a theory represented by a cellular spectrum (since everything in Topology is cellular), we hint in Subsection 4.4 at possible direction which could help with the "real" algebro-geometric problems.

Acknowledgements: First of all, I would like to thank A.Smirnov and I.Panin for many stimulating discussions since our 2004-2005 common stay at IAS. These really influenced my way of thinking about the subject. I want to thank O.Hauton, with whom we tried to produce the geometric construction of Symmetric operations for $p = 3$, and discussed various other related topics. Also, I'm very grateful to P.Brosnan, S.Gille, A.Kuznetsov, A.Lazarev, M.Levine, F.Morel, M.Rost, B.Totaro, V.Voevodsky, N.Yagita, S.Yagunov, and other people for many useful conversations. Special thanks to A.Lazarev and F.Morel for pointing me in the direction of the Quillen's paper [14]. Finally, I would like to express my gratitude to W.Wilson, whose book [24] gave me the inspiration for the current article. The support of EPSRC Responsive Mode grant EP/G032556/1 is gratefully acknowledged.

2 Algebraic Cobordism and other generalized oriented cohomology theories

2.1 Main definitions

Throughout the article k will denote the base field of characteristic 0. \mathbf{Sm}_k will denote the category of smooth quasi-projective varieties over k , and \mathbf{Sch}_k - the category of separated schemes of finite type over k . Let \mathbf{R}^* be the category of graded commutative rings.

Following M.Levine-F.Morel ([8, Definition 1.1.2]) and D.Quillen ([14]) we introduce the notion of the generalized oriented cohomology theory on \mathbf{Sm}_k . The only difference in comparison with [8, Definition 1.1.2] is that we impose the excision axiom (*EXCI*). All the "standard" theories, like Ω^* , CH^* and K_0 do satisfy this axiom, but not their \mathbb{A}^1 -analogues $MGL^{*,*}$, $H_{\mathcal{M}}^{*,*}$ and K_* . So, this axiom restricts our choice to the *pure* parts of \mathbb{A}^1 -theories: $A^{*,2*}$. And I should point out that this axiom (together with the axiom (*CONST*) below) is crucial for everything what follows - the main idea of the article is based on it.

Definition 2.1 (cf. [8, Definition 1.1.2]) *A generalized oriented cohomology theory on \mathbf{Sm}_k is given by:*

- (D1) *An additive (pull-back) functor $A^* : \mathbf{Sm}_k^{op} \rightarrow \mathbf{R}^*$.*
- (D2) *The structure of push forwards: for each projective morphism $f : Y \rightarrow X$ of relative dimension d , a homomorphism of graded $A^*(X)$ -modules:*

$$f_* : A^*(Y) \rightarrow A^*(X)$$

satisfying:

- (A1) *functoriality of push-forwards: $(Id_X)_* = Id_{A^*(X)}$, and*

For projective morphisms $f : Y \rightarrow X$, $g : Z \rightarrow Y$ of relative dimensions d and e ,

$$(f \circ g)_* = f_* \circ g_* : A^*(Z) \rightarrow A^{*+d+e}(X).$$

(A2) For pair of transversal morphisms $f : X \rightarrow Z$, $g : Y \rightarrow Z$ fitting into the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z, \end{array}$$

with f projective of relative dimension d ,

$$g^* f_* = f'_* g'^*.$$

(PB) For a rank n vector bundle $E \rightarrow X$ with the canonical quotient line bundle $\mathcal{O}(1) \rightarrow \mathbb{P}(E)$, the zero section $s : \mathbb{P}(E) \rightarrow \mathcal{O}(1)$, and $\xi \in A^1(\mathbb{P}(E))$ defined by

$$\xi := s^* s_*(1),$$

one has: $A^*(\mathbb{P}(E))$ is a free $A^*(X)$ -module with the basis

$$(1, \xi, \xi^2, \dots, \xi^{n-1}).$$

(EH) For a vector bundle $E \rightarrow X$ and an E -torsor $p : V \rightarrow X$ one has: $p^* : A^*(X) \rightarrow A^*(V)$ is an isomorphism.

(EXCI) For a smooth quasi-projective variety X with closed subscheme $Z \xrightarrow{i} X$ and open complement $U \xrightarrow{j} X$, one has an exact sequence:

$$A_*(Z) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \rightarrow 0,$$

where $A_*(Z) := \lim_{V \rightarrow Z} A_*(V)$ - the limit taken over all projective maps from smooth varieties to Z , and for a d -dimensional variety T , $A_*(T) := A^{d-*}(T)$.

Remark: 1) Notice, that (D2) contains the projection formula.

2) The extended homotopy property (EH) follows from the "usual" homotopy and (PB).

Whenever we refer to the generalized oriented cohomology theory in the sense of Definition 2.1, we will mean the theory satisfying the above set of axioms.

Quite often (especially, in our main results) we will need to impose an additional condition demanding our theory to be constant along field extensions. To formulate this condition, we need to remind that originally our theory was defined only for smooth quasi-projective varieties over k , in particular, for varieties of finite type. But one can extend it for localizations of such varieties by approximating them by finite-type ones. In particular, following M.Levine and F.Morel ([8, Subsection 4.4.1]), for any finitely generated field extension L/k we can define $A^*(L)$ as $\text{colim}_{U \subset X} A^*(U)$ where U runs over all open subsets of some smooth model X with $k(X) = L$ (recall, that we are in characteristic zero, so all field extensions are separable). Then we have the notion of a *generically constant* theory of M.Levine-F.Morel - see [8, Definition 4.4.1]. I will call such theories just *constant*.

(CONST) The theory is called "constant" if the natural map $A^*(k) \rightarrow A^*(L)$ is an isomorphism, for each finitely generated field extension L/k .

All the standard theories are constant, but it is very easy to construct a non-constant one.

Example 2.2 Let A^* be any theory (say, a constant one), and Y be a smooth quasi-projective variety over k . Then we can define a new theory: $A_{Y/k}^*(X) := A^*(Y \times_{\text{Spec}(k)} X)$. For example, we can take $Y = \text{Spec}(L)$, where L/k is a finite field extension. This theory will not be constant. For example, if L/k is Galois of degree n , then $A_{L/k}^*(\text{Spec}(L)) = \bigoplus_{i=1}^n A^*(\text{Spec}(L))$, while $A_{L/k}^*(\text{Spec}(k)) = A^*(\text{Spec}(L))$.

M. Levine and F. Morel constructed the universal generalized oriented cohomology theory Ω^* called Algebraic Cobordism (see [8, Theorem 1.2.6]). It has unique map to any other theory A^* . This theory satisfies $(CONST)$. Being an analogue of complex-oriented cobordism theory MU^{2*} in topology, (for fields possessing an embedding to \mathbb{C}) it has a *topological realization functor* $\Omega^*(X) \rightarrow MU^{2*}(X(\mathbb{C}))$, which is an isomorphism for $X = \text{Spec}(k)$.

2.2 An associated Borel-Moore theory

Each generalised oriented cohomology theory on \mathbf{Sm}_k can be extended to the Borel-Moore functor on \mathbf{Sch}_k in the sense of [8, Definition 2.1.2] - see [8, Remark 2.1.4]. We will not need most of the features of such a functor, only the push-forward maps which are completely straightforward, so will not list it's axioms here. Later, in Subsection 4.3, in the case of *theories of rational type* we will need the *refined pull-backs*, but these will be supplied by the Algebraic Cobordism case done by M. Levine and F. Morel.

Definition 2.3 For a quasi-projective scheme Z define $A_*(Z) := \lim_{V \rightarrow Z} A_*(V)$, where the limit is taken over all $v : V \rightarrow Z$ with v projective and V smooth.

Clearly, $A_*(Z) = A_*(Z_{red})$, and if $Z = \bigcup_{i=1}^m Z_i$ is the decomposition into irreducible components, then we have an exact sequence:

$$0 \longleftarrow A_*(Z) \longleftarrow \bigoplus_{i=1}^m A_*(Z_i) \longleftarrow \bigoplus_{i,j=1}^m A_*(Z_i \cap Z_j).$$

More generally, for a closed embedding $S \subset Z$ with the open compliment U , we have an excision sequence:

$$0 \longleftarrow A_*(U) \longleftarrow A_*(Z) \longleftarrow A_*(S).$$

A priori, $A_*(Z)$ for a singular scheme Z is expressed in terms of A_* of infinitely many smooth schemes. But Proposition 7.7 shows that one can construct a "finite" presentation related to the resolution of singularities.

2.3 Formal group law

Any theory in the sense of Definition 2.1 (even without $(EXCI)$) has Chern classes. Namely, if \mathcal{V} is a vector bundle of dimension d on X , then $\xi \in A^1(\mathbb{P}_X(\mathcal{V}^\vee))$ (as in the axiom (PB)) satisfies the unique equation:

$$\sum_{i=0}^d (-1)^i c_i^A(\mathcal{V}) \cdot \xi^{d-i} = 0,$$

where $c_0^A(\mathcal{V}) = 1$, and $c_i^A(\mathcal{V}) \in A^i(X)$ are some elements. These satisfy the usual Cartan formula, and in the case of a linear bundle \mathcal{L} , $c_1^A(\mathcal{L}) = s^* s_*(1)$, where $s : X \rightarrow \mathcal{L}$ is a zero section. By [8, Theorem 2.3.13], any theory A^* as above satisfies the axiom:

(DIM) For any line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ on a smooth X of dimension $< n$, one has: $c_1^A(\mathcal{L}_1) \cdot \dots \cdot c_1^A(\mathcal{L}_n) = 0 \in A_*(X)$.

Thus, any power series in Chern classes can be evaluated on any element of $A_*(X)$.

To any theory A^* as above one can associate the Formal Group Law (FGL, for short) (A, F_A) , where A is the coefficient ring of A^* , and

$$F_A(x, y) = \text{Segre}^*(t) \in A[[x, y]] = A^*(\mathbb{P}^\infty \times \mathbb{P}^\infty),$$

where $\mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{\text{Segre}} \mathbb{P}^\infty$ is the Segre embedding, and x, y, t are the 1-st Chern classes of $\mathcal{O}(1)$ of the respective copies of \mathbb{P}^∞ . We will denote the coefficients of F_A as $a_{i,j}^A$. Thus,

$$F_A(x, y) = \sum_{i,j} a_{i,j}^A \cdot x^i \cdot y^j.$$

The formal group law describes how to compute the 1-st Chern class of a tensor product of two line bundles in terms of the 1-st Chern classes of factors:

$$c_1^A(\mathcal{L} \otimes \mathcal{M}) = F_A(c_1^A(\mathcal{L}), c_1^A(\mathcal{M})).$$

The universal formal group law (\mathbb{L}, F_U) has canonical morphism to any other formal group law, in particular, to (A, F_A) . M. Levine and F. Morel have shown that, in the case of algebraic cobordism, the respective map is an isomorphism - see [8, Theorem 1.2.7]. In particular, $\Omega^*(k) = \mathbb{L}^*$, for any field k .

We call the theory A^* *additive*, if it's FGL is additive. By the result of M. Levine-F. Morel - see [8, Theorem 1.2.2], the Chow groups CH^* is the universal additive (= *ordinary*) theory.

3 Operations

3.1 Category **SmOp**

As in topology, operation is just a morphism of theories considered as contravariant functors on **Sm_k** (thus, commuting with the pull-backs). Such operations appear to be of "various quality", and the best behaving of them are, so-called, *stable operations*. Most of the operations used are stable (as exceptions I can only recall Adams operations in K-theory, and Symmetric operations in Algebraic Cobordism (see [19])). Although, the stable operations are simple, and it is easy to construct them, they are less subtle than the *unstable* ones. The aim of the current article is to develop an effective method of producing unstable operations. And, although, in the end, stable operations is not what we are after (there are more or less no questions left about them), they provide an important "coordinate system" in which one can study unstable ones. To be able to talk about "stability" we need to introduce some notion of *suspension*. Let me remind that the typical example of our theory A^* is the pure part $A^{2*,*}$ of some \mathbb{A}^1 -theory. In particular, we do not have *simplicial* tools at our disposal. As an easy way out of it (following V. Voevodsky and I. Panin-A. Smirnov), let me introduce:

Definition 3.1 *Category **SmOp** has objects (X, U) , where X is a smooth quasi-projective variety over k , and $U \xrightarrow{i} X$ is an open subvariety. Morphisms from (X, U) to (Y, V) are maps $X \xrightarrow{f} Y$ which map U to V . We have a natural functor:*

$$N : \mathbf{Sm}_k \longrightarrow \mathbf{SmOp},$$

sending X to (X, \emptyset) .

In **SmOp** we can define Smash-product by the formula:

$$(X, U) \wedge (Y, V) := (X \times Y, X \times V \cup U \times Y),$$

which permits to introduce the suspension:

Definition 3.2

$$\Sigma_T(X, U) := (X, U) \wedge (\mathbb{P}^1, \mathbb{P}^1 \setminus 0).$$

Any theory A^* in the sense of Definition 2.1 can be extended to a contravariant functor $A^* : \mathbf{SmOp} \rightarrow \mathbf{Ab}$ as follows:

$$A^*((X, U)) := \text{Ker}(A^*(X) \xrightarrow{i^*} A^*(U)),$$

with the pull-backs naturally induced by those in \mathbf{Sm}_k . We have an external product:

$$A^*((X, U)) \otimes A^*((Y, V)) \xrightarrow{\wedge} A^*((X, U) \wedge (Y, V)),$$

and a canonical element $\varepsilon^A = c_1^A(\mathcal{O}(1)) \in A^1((\mathbb{P}^1, \mathbb{P}^1 \setminus 0))$ - the class of a rational point. We get the natural isomorphism:

$$\begin{aligned} \Sigma_T : A^n((X, U)) &\xrightarrow{=} A^{n+1}(\Sigma_T(X, U)) \\ x &\mapsto x \wedge \varepsilon^A. \end{aligned}$$

Definition 3.3 Let A^* and B^* be theories in the sense of Definition 2.1. An operation $G : A^n \rightarrow B^m$ is a morphism of contravariant functors of sets pointed by 0 (in other words, a transformation commuting with the pull-backs, and sending zero to zero). An operation is called additive, if it is a homomorphism of abelian group.

Note, that such an operation extends uniquely to a morphism of contravariant functors on \mathbf{SmOp} . Moreover, the condition $0 \mapsto 0$ is equivalent to the existence of such an extension (since $A^*((X, X)) = 0$, and there exists morphism $(X, U) \rightarrow (X, X)$).

Definition 3.4 A stable operation $G : A^* \rightarrow B^{*+l}$ is a set of operations $\{G^n : A^n \rightarrow B^{n+l}, n \in \mathbb{Z}\}$, which commute with Σ_T .

As one would expect,

Proposition 3.5 Any stable operation is additive.

Proof:

Lemma 3.6 Let $\alpha, \beta, \gamma : (\mathbb{P}^1, \mathbb{P}^1 \setminus 0) \rightarrow (\mathbb{P}^1, (\mathbb{P}^1 \setminus 0))^{\times 2}$ be defined as follows: $\alpha = id \times \infty$, $\beta = \infty \times id$, $\gamma = \Delta$. Let C^* be any theory in the sense of Definition 2.1. Then, for any $R \in \text{Ob}(\mathbf{SmOp})$, and $\delta_R = \delta \wedge id_R$, we have: $\gamma_R^* = \alpha_R^* + \beta_R^*$.

Proof: Let $\varepsilon = \varepsilon^C$. Then we have: $C^*((\mathbb{P}^1, (\mathbb{P}^1 \setminus 0))^{\times 2} \wedge R) = \varepsilon_1 \cdot C^*(R) \oplus \varepsilon_2 \cdot C^*(R) \oplus \varepsilon_1 \cdot \varepsilon_2 \cdot C^*(R)$. Clearly, $\alpha^*, \beta^*, \gamma^*$ are zero on $\varepsilon_1 \cdot \varepsilon_2$, while $\alpha^*(\varepsilon_1) = \varepsilon$, $\alpha^*(\varepsilon_2) = 0$, $\beta^*(\varepsilon_1) = 0$, $\beta^*(\varepsilon_2) = \varepsilon$, and $\gamma^*(\varepsilon_1) = \gamma^*(\varepsilon_2) = \varepsilon$. \square

Denote: $\varepsilon_1 \cdot x + \varepsilon_2 \cdot y + \varepsilon_1 \cdot \varepsilon_2 \cdot 0 \in A^*((\mathbb{P}^1, (\mathbb{P}^1 \setminus 0))^{\times 2} \wedge R)$ as $(x, y, 0)$. Then

$$G(\Sigma_T(x + y)) = G(\gamma^*(x, y, 0)) = \gamma^*G(x, y, 0) = (\alpha^* + \beta^*)G(x, y, 0) = G(\Sigma_T x) + G(\Sigma_T y),$$

and since G is stable, $G(x + y) = G(x) + G(y)$. \square

Definition 3.7 An operation $G : A^* \rightarrow B^*$ is multiplicative if, for each X , the respective transformation is a homomorphism of rings.

To each multiplicative operation one can assign certain power series - the *inverse Todd genus* $\gamma_G = b_0 \cdot x + b_1 \cdot x^2 + b_2 \cdot x^3 + \dots \in B[[x]]$, where, for $x^A = c_1^A(\mathcal{O}(1))$, $x^B = c_1^B(\mathcal{O}(1))$, one has: $G(x^A) = \gamma_G(x^B) \in B[[x^B]] = B(\mathbb{P}^\infty)$. Also, we have $\varphi_G : A \rightarrow B$ - the homomorphism of coefficient rings. The pair (φ_G, γ_G) is a morphism of formal group laws: $(A, F_A) \longrightarrow (B, F_B)$. In other words,

$$\varphi_G(F_A)(\gamma_G(u), \gamma_G(v)) = \gamma_G(F_B(u, v)).$$

Of course, the composition of multiplicative operations corresponds to the composition of morphisms of formal group laws:

$$(\varphi_{H \circ G}, \gamma_{H \circ G}(x)) = (\varphi_H \circ \varphi_G, \varphi_H(\gamma_G)(\gamma_H(x))).$$

In the case of $A^* = \Omega^*$, and b_0 invertible in B , the homomorphism φ_G is completely determined by γ_G . Namely, \mathbb{L} is generated as a ring by universal coefficients $a_{i,j}^\Omega$, and $\varphi_G(a_{i,j}^\Omega)$ is the respective coefficient of the formal group law $F_B^{\gamma_G}(x, y) = \gamma_G(F_B(\gamma_G^{-1}(x), \gamma_G^{-1}(y)))$. Moreover, one has:

Theorem 3.8 (Panin-Smirnov+Levine-Morel) *If b_0 is invertible in B , then for each $\gamma = b_0x + b_1x^2 + b_2x^3 + \dots \in B[[x]]$, there exists unique multiplicative operation $G : \Omega^* \rightarrow B^*$ with such γ_G .*

Proof: I. Panin and A. Smirnov have shown in [12] (see also [8] and [6]) that any reparametrization as above gives rise to the change of orientation (push-forward structure) on B^* , while M. Levine and F. Morel have proven universality of Ω^* ([8, Theorem 1.2.6]), which gives the morphism of theories $\Omega^* \rightarrow \tilde{B}^*$, which provides the needed operation, since the pull-back structure on B^* and \tilde{B}^* is the same. \square

Below we will be able to generalize this result substantially - see Theorems 6.9 and 6.13.

The following statement describes the relation between stable and multiplicative operations.

Proposition 3.9 *Let $G : A^* \rightarrow B^*$ be a multiplicative operation with $\gamma_G = b_0x + b_1x^2 + \dots$. Then G is stable if and only if $b_0 = 1$.*

Proof: Since G is multiplicative on \mathbf{Sm}_k , it will be multiplicative on \mathbf{SmOp} (w.r.to \wedge). Then

$$G(\Sigma_T x) = G(x \wedge \varepsilon^A) = G(x) \wedge G(\varepsilon^A) = G(x) \wedge (b_0 \varepsilon^B) = b_0 \Sigma_T(G(x)).$$

Thus we get the needed identity (for all x) if and only if $b_0 = 1$. \square

Example 3.10 *Let $S_{L-N}^{Tot} : \Omega^* \longrightarrow \Omega^*[b_1, b_2, \dots] = \Omega^*[\bar{b}]$ be the Total Landweber-Novikov operation. It is the multiplicative operation corresponding to the power series $x + b_1x^2 + b_2x^3 + \dots$, where b_i are independent variables (see [8, Example 4.1.25] and [14]). By Proposition 3.9 this operation is stable.*

Any stable multiplicative operation $G : \Omega^* \rightarrow B^*$ is a specialization of S_{L-N}^{Tot} . Namely, for each such G there exists unique morphism of theories $\theta_G : \Omega^*[b_1, b_2, \dots] \rightarrow B^*$ such that $G = \theta_G \circ S_{L-N}^{Tot}$. This θ_G is the canonical morphism of theories on Ω^* , and sends b_i 's to the coefficients of γ_G .

3.2 Stable operations in Algebraic Cobordism

We already have some examples of stable operations in Ω^* - the components of the Total Landweber-Novikov operation S_{L-N}^{Tot} . It appears that, basically, there is nothing else out there (as in topological case).

Theorem 3.11 *There exists natural 1-to-1 correspondence between the set $\text{Hom}_{\mathbb{L}}(\mathbb{L}[\bar{b}], \mathbb{L})$, and the set of stable operations $\Omega^* \rightarrow \Omega^*$ given by: $\psi \leftrightarrow G_\psi$, where G_ψ is the composition: $\Omega^* \xrightarrow{S_{L-N}^{Tot}} \Omega^*[\bar{b}] \xrightarrow{\otimes \psi} \Omega^*$.*

Proof: Since S_{L-N}^{Tot} and $\otimes\psi$ are stable operations, so is their composition. Let now G be some stable operation $\Omega^* \rightarrow \Omega^*$. Then G is additive. In particular, $G|_{\mathbb{L}}$ is additive homomorphism $\mathbb{L} \rightarrow \mathbb{L}$. Consider the commutative diagram:

$$\begin{array}{ccc} \mathbb{L} & \xrightarrow{S_{L-N}^{Tot}} & \mathbb{L}[\bar{b}] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\bar{d}] & \xrightarrow{S} & \mathbb{Z}[\bar{d}][\bar{b}], \end{array}$$

where the vertical maps are induced by the natural embedding of rings $\mathbb{L} \hookrightarrow \mathbb{Z}[d_1, d_2, \dots]$ corresponding to the twist of the additive formal group law by the change of parameter: $\delta(y) = y + d_1 y^2 + d_2 y^3 + \dots$, and S maps d_i to e_i - the coefficient of the power series $\rho(y) = \beta(\delta(y))$, where $\beta(x) = x + b_1 x^2 + b_2 x^3 + \dots$. In particular, the vertical maps are isomorphisms $\otimes \mathbb{Q}$,

$$\mathbb{Z}[\bar{d}][\bar{b}] = \mathbb{Z}[\bar{d}][\bar{e}] \quad \text{and} \quad \mathbb{L}[\bar{b}] \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{L}[\bar{e}] \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Thus, for our \mathbb{Z} -linear map $\mathbb{L} \xrightarrow{G|_{\mathbb{L}}} \mathbb{L}$ there exists unique \mathbb{L} -linear map $\mathbb{L}[\bar{b}] \xrightarrow{\psi_G} \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the composition

$$\mathbb{L} \xrightarrow{S_{L-N}^{Tot}} \mathbb{L}[\bar{b}] \xrightarrow{\psi_G} \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$$

coincides with $G|_{\mathbb{L}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider the operation:

$$H = G - \psi_G \circ S_{L-N}^{Tot} : \Omega^* \longrightarrow \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Let us show that $H = 0$.

Lemma 3.12 *Let $A^* \xrightarrow{H} B^*$ be stable operation such that $H|_X = 0$. Then $H|_{X \times \mathbb{P}^1} = 0$.*

Proof: The maps $\mathbb{P}^1 \rightleftarrows \text{Spec}(k)$ define the decomposition: $C^*(X \times \mathbb{P}^1) = C^*(X) \oplus C^*(\Sigma X)$ respected by additive operations. Moreover, since H is stable, and $H|_X$ is zero, so is $H|_{\Sigma X}$. Hence, $H|_{X \times \mathbb{P}^1} = 0$. \square

Lemma 3.13 *Suppose, B has no torsion, and $A^* \xrightarrow{H} B^*$ is stable operation such that $H|_{\text{Spec}(k)} = 0$. Then $H|_{(\mathbb{P}^\infty) \times r} = 0$, for any r .*

Proof: We need to show that $H|_{(\mathbb{P}^N) \times r} = 0$, for all N and r . Consider the natural projection $p : ((\mathbb{P}^1)^{\times N})^{\times r} \rightarrow (\mathbb{P}^N)^{\times r}$. Since $p^* : B^*((\mathbb{P}^N)^{\times r}) \rightarrow B^*((\mathbb{P}^1)^{\times N})^{\times r}$ is injective, by Lemma 3.12, $H|_{(\mathbb{P}^\infty) \times r} = 0$. \square

Remark 3.14 *The condition that B has no torsion is essential. Take, for example $A^* = B^* = \text{CH}^*/2$, and $H = G_1 - G_2$, where $G_1 = \text{id}$ with $\gamma_{G_1} = x$ and $G_2 = \text{St}^{Tot}$ with $\gamma_{G_2} = x + x^2$ - the Total Steenrod operation. Then $\varphi_{G_1} = \varphi_{G_2}$ since there exists only one homomorphism of rings $\mathbb{Z}/2 \rightarrow \mathbb{Z}/2$, and so $H|_{\text{Spec}(k)=0}$. At the same time, clearly, $H|_{\mathbb{P}^\infty} \neq 0$.*

Proposition 3.15 *Let A^* satisfies (CONST), and B^* be any theory in the sense of Definition 2.1. Let $A^* \xrightarrow{H} B^*$ be an additive operation (not necessarily stable!) such that $H|_{(\mathbb{P}^\infty) \times r} = 0$, for any r . Then $H = 0$.*

Proof: Let us prove by induction on the dimension of X that $H|_{X \times (\mathbb{P}^\infty)^{\times r}} = 0$, for all r . The base ($\dim(X) = 0$) follows from our conditions. Suppose $\dim(X) = d$, and the statement is known for varieties of smaller dimension. Since $A^*(X \times (\mathbb{P}^N)^{\times r})$ is a free module over $A^*(X)$ with basis consisting of monomials $\bar{\xi}^{\bar{m}} = \prod_{i=1}^r \xi_i^{m_i}$, where $\xi_i = c_1^A(\mathcal{O}(1)_i)$, it is sufficient to prove that $H(x \cdot \bar{\xi}^{\bar{m}}) = 0$, for any $x \in A^*(X)$, for any \bar{m} . Because A^* satisfies *(CONST)*, we have: $H(x|_{\text{Spec}(k(X))} \cdot \bar{\xi}^{\bar{m}}) = 0$, and by additivity of H we can assume that $x|_{\text{Spec}(k(X))} = 0$, that is, x is supported on some closed subvariety $Y \subset X$ (here we use *(EXCI)*). By the result of Hironaka (see Theorem 8.2), there exists a permitted blow up $\pi : \tilde{X} \rightarrow X$ with centers over Y and of dimension $< \dim(Y)$, such that the proper preimage \tilde{Y} of Y is smooth. Since $\pi^* : B^*(X) \rightarrow B^*(\tilde{X})$ is injective, it is sufficient to show that $H(\pi^*(x) \cdot \bar{\xi}^{\bar{m}}) = 0$. We have:

$$\begin{aligned}\tilde{X} &= X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}^{-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X \\ \tilde{Y} &= Y_n \xrightarrow{\pi'_n} Y_{n-1} \xrightarrow{\pi'_{n-1}^{-1}} \dots \xrightarrow{\pi'_2} Y_1 \xrightarrow{\pi'_1} Y_0 = Y,\end{aligned}$$

where $X_{i+1} = \text{Bl}_{Z_i \subset X_i}$, $Z_i \subset X_i$ is smooth of dimension $< \dim(Y)$. Let $y_i \in A^*(X_i)$ be some element with support on Y_i . Then $\pi_i^*(y_i) = y_{i+1} + u_{i+1}$, where y_{i+1} has support in Y_{i+1} and u_{i+1} has support in the special divisor $\mathbb{P}_{Z_i}(\mathcal{N}_{Z_i \rightarrow X_i})$.

Lemma 3.16 *Let $A^* \xrightarrow{H} B^*$ be an additive operation, $u \in A^*(Z)$, and $X \xleftarrow{f} Z \xrightarrow{g} \mathbb{P}_Z(\mathcal{N}_f \oplus \mathcal{O})$ be regular embeddings. Then*

$$\{H(g_*(u)) = 0\} \Rightarrow \{H(f_*(u)) = 0\}.$$

Proof: Let us use the deformation to the normal cone construction. We have varieties $\tilde{W} = \text{Bl}_{Z \times \{0\} \subset X \times \mathbb{A}^1}$, $\tilde{Z} = Z \times \mathbb{A}^1$, $W_0 = \mathbb{P}_Z(\mathcal{N}_f \oplus \mathcal{O})$, $W_1 = X \times \{1\}$, fitting into the diagram:

$$\begin{array}{ccccc} W_0 & \xrightarrow{i_0} & \tilde{W} & \xleftarrow{i_1} & W_1 \\ g \uparrow & & h \uparrow & & \uparrow f \\ Z & \xrightarrow{j_0} & \tilde{Z} & \xleftarrow{j_1} & Z \end{array}$$

with both squares transversal cartesian. Let $\tilde{Z} \xrightarrow{p} Z$ be the natural projection. Since B^* satisfies *(EXCI)*, $H(h_*p^*(u))$ has support in \tilde{Z} . That is, $H(h_*p^*(u)) = h_*(v)$, for some $v \in B^*(\tilde{Z})$. Then $i_0^*H(h_*p^*(u)) = H(i_0^*h_*p^*(u)) = H(g_*j_0^*p^*(u)) = H(g_*(u)) = 0$ should be equal to $i_0^*h_*(v) = g_*j_0^*(v)$. But j_0^* is an isomorphism, and g_* is an injection. Hence, $v = 0$, and so $H(h_*p^*(u)) = 0$. This implies that: $0 = i_1^*H(h_*p^*(u)) = H(i_1^*h_*p^*(u)) = H(f_*j_1^*p^*(u)) = H(f_*(u))$. \square

Lemma 3.17 *Let \mathcal{V} be vector bundle on Z , and $V = \mathbb{P}_Z(\mathcal{V})$. Let $A^* \xrightarrow{H} B^*$ be additive operation s.t. $H|_{Z \times (\mathbb{P}^\infty)^{\times r}} = 0$, $\forall r$. Then $H|_{V \times (\mathbb{P}^\infty)^{\times r}} = 0$, $\forall r$.*

Proof: $A^*(V)$ as an $A^*(Z)$ -module is generated by powers of $c_1^A(\mathcal{O}(1))$. There are very ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ on V such that $\mathcal{O}(1) = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$. Hence, any element in $A^*(V)$ can be written as an $A^*(Z)$ -linear combination of $c_1^A(\mathcal{L}_1)^{m_1} \cdot c_1^A(\mathcal{L}_2)^{m_2}$. And each such element is a pull-back of a certain element from $A^*(Z \times (\mathbb{P}^\infty)^{\times 2})$. Thus, any element from $A^*(V \times (\mathbb{P}^N)^{\times r})$ is a sum of elements pulled back from $A^*(Z \times (\mathbb{P}^M)^{\times r+2})$, and so H must be trivial on it. \square

Lemma 3.18 *Let $Z \xrightarrow{f} T$ be a regular embedding (of smooth varieties). Then $\{H|_{Z \times (\mathbb{P}^\infty)^{\times r}} = 0, \text{ for any } r\} \Rightarrow H \text{ is zero on the image}((f \times \text{id})_*) \subset A^*(T \times (\mathbb{P}^\infty)^{\times r}), \text{ for any } r.$*

Proof: Follows immediately from Lemmas 3.16 and 3.17. \square

Since u_{i+1} has support on a smooth subvariety $\mathbb{P}_{Z_i}(\mathcal{N}_{Z_i \subset X_i})$, it follows from the inductive assumption and Lemma 3.18 that $H(u_{i+1} \cdot \bar{\xi}^{\bar{m}}) = 0$, and the same will be true when we will lift u_{i+1} to \tilde{X} . Take now $y_0 = x$, and construct the elements y_i, u_i as above. Then $\tilde{y} = y_n$ has support in \tilde{Y} , and by the above, $H(\tilde{y} \cdot \bar{\xi}^{\bar{m}}) = H(\pi^*(x) \cdot \bar{\xi}^{\bar{m}})$. Thus, we can reduce to the case where x has support on a smooth subvariety $Y \subset X$, where it follows from the inductive assumption and Lemma 3.18. Induction step is done, and Proposition 3.15 is proven. \square

We have proven that the composition $\Omega^* \xrightarrow{G} \Omega^* \hookrightarrow \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with the composition $\Omega^* \xrightarrow{S_{L-N}^{Tot}} \Omega^*[\bar{b}] \xrightarrow{\psi_G} \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}$, that is G is a linear combination (infinite, in general) of the Landweber-Novikov operations. It remains to show that, in reality, this linear combination has coefficients in \mathbb{L} . Induction on the degree of the monomial $\bar{b}^{\bar{r}} = \prod_i b_i^{r_i}$, where $\bar{r} = (r_1, r_2, \dots)$, and $\deg(b_i) = i$. The base: the coefficient at $\bar{b}^{\bar{0}} = 1$ is $G(1) \in \mathbb{L}$. Suppose, the coefficients at all monomial of smaller degrees are in \mathbb{L} . Consider $X = \times_i (\mathbb{P}^{i+1})^{\times r_i}$, and $x = \times_i (h)^{\times r_i} \in \Omega^{\sum_i r_i}(X)$, where $h = c_1^\Omega(\mathcal{O}(1))$. The action of the operation $S_{L-N}^{\bar{r}}$ on the element $[v : V \rightarrow X]$ can be computed as follows: let $\lambda_1, \lambda_2, \dots$ be roots of $(-T_V + v^*T_X)$. Then

$$S_{L-N}^{\bar{r}}([v]) = \text{coefficient at } \bar{b}^{\bar{r}} \text{ in } v_*(\prod_j (1 + \lambda_j b_1 + \lambda_j^2 b_2 + \dots)(1_V))$$

In our case, $(-T_V + v^*T_X) = \oplus \mathcal{O}(1)$ - one summand per each copy of \mathbb{P}^{i+1} , for all i . Hence, $S_{L-N}^{\bar{s}}(x) = 0$, for all \bar{s} of degree $= \deg(\bar{r})$ different from \bar{r} , while $S_{L-N}^{\bar{r}}(x) = [\text{Spec}(k) \hookrightarrow X] = pt$ - the class of a rational point. By dimensional reasons, the operations corresponding to monomials of larger degrees also vanish on x . Hence, $G(x) = \mu_{\bar{r}} \cdot pt + \sum_{\deg(\bar{s}) < \deg(\bar{r})} \mu_{\bar{s}} \cdot S_{L-N}^{\bar{s}}(x)$. But $G(x) \in \Omega^*(X)$, and all the summands aside from the first one belong to $\Omega^*(X)$ by the inductive assumption. Hence, $\mu_{\bar{r}} \cdot pt \in \Omega^*(X)$ which implies that $\mu_{\bar{r}} \in \mathbb{L}$. Induction step is proven, and so the coefficients belong to \mathbb{L} . It remains to run the Lemmas 3.12, 3.13 and Proposition 3.15 again (for \mathbb{L} -linear combination, this time), and Theorem 3.11 is proven. \square

3.3 Unstable operations in Algebraic Cobordism (uniqueness)

Unstable operations can be described in terms of stable ones. In analogy with topology we have:

Theorem 3.19 *The correspondence: $G_\psi \leftrightarrow \psi$, where $(G_\psi)_{\mathbb{Q}}$ is the map: $\Omega^* \xrightarrow{S_{L-N}^{Tot}} \Omega^*[\bar{b}] \xrightarrow{\otimes \psi} \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}$ identifies the set of (unstable) additive operations $\Omega^n \rightarrow \Omega^m$ with the subset of $\text{Hom}_{\mathbb{L}}(\mathbb{L}[\bar{b}], \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q})_{(m-n)}$. In other words, any such operations can be interpreted as a unique $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ -linear combination (infinite, in general) of the Landweber-Novikov operations.*

Proof: By Proposition 3.15 we know that any additive operation $\Omega^n \xrightarrow{G} \Omega^m$ is completely defined by its action on $(\mathbb{P}^\infty)^{\times r}$, for all r . Thus, it is sufficient to show that there exists such $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ -linear combination of the Landweber-Novikov operations which coincides with $G_{\mathbb{Q}}$ when restricted to $(\mathbb{P}^\infty)^{\times r}$, $\forall r$. We have mutually inverse operations:

$$\Omega^* \otimes_{\mathbb{Z}} \mathbb{Q} \xrightleftharpoons[\beta]{\alpha} \text{CH}^* \otimes_{\mathbb{Z}} \mathbb{Q}[\bar{d}],$$

where $\gamma_\alpha^{-1} = x + d_1x^2 + d_2x^3 + \dots = \varphi_\alpha(\log \Omega)$, and $\gamma_\beta = \log \Omega$. Thus, we obtain a commutative diagram:

$$\begin{array}{ccc} \Omega^n \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{G_{\mathbb{Q}}} & \Omega^m \otimes_{\mathbb{Z}} \mathbb{Q} \\ \beta \uparrow & & \downarrow \alpha \\ (\mathrm{CH}^* \otimes_{\mathbb{Z}} \mathbb{Q}[\bar{d}])_{(n)} & \xrightarrow{H} & (\mathrm{CH}^* \otimes_{\mathbb{Z}} \mathbb{Q}[\bar{d}])_{(m)}, \end{array}$$

where H is an additive operation between additive theories.

Let A^* and B^* be two theories in the sense of Definition 2.1. Let $x_i = c_1^A(\mathcal{O}(1)_i)$, and $y_i = c_1^B(\mathcal{O}(1)_i)$. Then $A^*((\mathbb{P}^\infty)^{\times r})$ is a free $A^*(\mathrm{Spec}(k))$ -module with the basis \bar{x}_I , and $B^*((\mathbb{P}^\infty)^{\times r})$ is a free $B^*(\mathrm{Spec}(k))$ -module with the basis \bar{y}_I .

Lemma 3.20 *Let $A^n \xrightarrow{H} B^m$ be an additive operation of additive theories. Suppose B has no torsion. Then there exists a homomorphism of abelian groups $A^*(\mathrm{Spec}(k)) \xrightarrow{\tilde{H}} B^{*+m-n}(\mathrm{Spec}(k))$ such that $H(u \cdot \bar{x}_I) = \tilde{H}(u) \cdot \bar{y}_I$, for all I and all $u \in A^{n-\deg(I)}(\mathrm{Spec}(k))$.*

Proof: Because of the partial diagonals, it is sufficient to treat the case $\bar{x}_I = x_1 \cdot x_2 \cdot \dots \cdot x_r$. Thus, in any degree we have just one such monomial, and we only need to show that $H(u \cdot \bar{x}_I)$ is \bar{y}_I times the linear function on u . Changing our A^* and B^* by $(A')^* = A^*[[x_1, \dots, \hat{x}_i, \dots, x_r]]$, and $(B')^* = B^*[[x_1, \dots, \hat{x}_i, \dots, x_r]]$, we can assume that $k = 1$. Consider the Segre embedding $\mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{f} \mathbb{P}^\infty$. Then $f^*(u \cdot x) = u \cdot x_1 + u \cdot x_2$. Let $H(u \cdot x) = \gamma(y) = \gamma_0 + \gamma_1 y + \gamma_2 y^2 + \dots \in B^*[[y]]$. Restricting to $\mathrm{Spec}(k) \hookrightarrow \mathbb{P}^\infty$, we see that $\gamma_0 = 0$. Let $\gamma(y) = \gamma_1 \cdot y + \gamma_s \cdot y^s + \dots$, that is, the next after the linear non-zero term has degree s . Then from the equality: $f^*(H(u \cdot x)) = H(f^*(u \cdot x))$, we get:

$$\gamma(y_1 + y_2) = \gamma(y_1) + \gamma(y_2).$$

Comparing coefficients at $y_1 \cdot y_2^{s-1}$, we get: $s \cdot \gamma_s = 0$. Since B has no torsion, we get that $\gamma(y) = \gamma_1 \cdot y$ is linear. Thus, we have shown that $H(u \cdot (x_1 \cdot \dots \cdot x_r)) = v \cdot (y_1 \cdot \dots \cdot y_r)$, and the correspondence $u \mapsto v$ defines an additive map $A^{n-r}(\mathrm{Spec}(k)) \xrightarrow{\tilde{H}} B^{m-r}(\mathrm{Spec}(k))$. \square

The map $\mathbb{L} \rightarrow \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\beta \circ \tilde{H} \circ \alpha|_{\mathrm{Spec}(k)}} \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ is additive. As we saw in the proof of Theorem 3.11, this map can be presented as the composition: $\mathbb{L} \xrightarrow{S_{L-N}^{Tot}} \mathbb{L}[\bar{b}] \xrightarrow{\otimes \psi} \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$, for some $\psi \in \mathrm{Hom}_{\mathbb{L}}(\mathbb{L}[\bar{b}], \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q})_{(m-n)}$. Then Lemma 3.20 shows that on $(\mathbb{P}^\infty)^{\times r}$, for all r , H coincides with the above $\mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ -linear combination of the Landweber-Novikov operations. \square

The natural question arises: which rational linear combinations of Landweber-Novikov operations are realized as (unstable) operations $\Omega^n \rightarrow \Omega^m$? It appears that exactly those which take integral values on $\Omega^n((\mathbb{P}^\infty)^{\times r})$, for all r - see Theorem 6.1. But this result is much more difficult than everything we discussed so far, and we will need the inductive description of the Algebraic Cobordism theory and various new tools in order to prove it.

4 Theories of rational type

In order to work with unstable operations we will need to produce the description of our theories which is inductive on the dimension. Not all theories admit a satisfactory description of the type we want. And the ones which do will be called theories of *rational type*. Later we will see that these are exactly the

free theories of M.Levine-F.Morel. The needed description of the theory will be obtained in stages. The one which is actually used is provided by the bi-complex^{*} \mathfrak{c} , but to get there we will need to introduce bi-complexes^{*} \mathfrak{a} and \mathfrak{b} , and to show that the Algebraic Cobordism of M.Levine-F.Morel is a theory of *rational type*.

4.1 The bi-complex^{*} \mathfrak{a}

Let A^* be some theory (in the sense of Definition 2.1). Let X be smooth irreducible variety over k . Consider the following bi-complex $\mathfrak{a} = \mathfrak{a}(A^*)$:

$$\begin{array}{ccc} a_{1,0} & \xrightarrow{d_{1,0}} & a_{0,0} \\ & \uparrow d_{0,1} & \\ & a_{0,1} & \end{array}$$

(such a "small" bi-complex will be called a *bi-complex^{*}* below), where

- $a_{0,0} := \bigoplus_{V \rightarrow X} A_*(V)$, where the sum is taken over all projective maps $V \rightarrow X$, such that V is smooth and $\dim(V) < \dim(X)$;
- $a_{1,0} := \bigoplus_{V' \rightarrow V \rightarrow X} A_*(V')$, where the sum is taken over all projective composable maps $V' \xrightarrow{\pi} V \rightarrow X$ such that V and V' are smooth, and $\dim(V) < \dim(X)$, $\dim(V') < \dim(X)$;
- $a_{0,1} := \bigoplus_{W \rightarrow X \times \mathbb{P}^1} A_{*+1}(W)$, where the sum is taken over all projective maps $w : W \rightarrow X \times \mathbb{P}^1$ with W smooth, $\dim(W) \leq \dim(X)$, and $W_0 = w^{-1}(X \times \{0\}) \xrightarrow{i_0} W$, $W_1 = w^{-1}(X \times \{1\}) \xrightarrow{i_1} W$ are divisors with normal crossing.

and the differentials are defined as follows:

- $d_{1,0}((V' \xrightarrow{\pi} V, y)) = (V \xrightarrow{v} X, \pi_*(y)) - (V' \xrightarrow{v'} X, y)$;
- $d_{0,1}((W \xrightarrow{w} X \times \mathbb{P}^1, z)) = (W_1 \xrightarrow{w_1} X, i_1^*(z)) - (W_0 \xrightarrow{w_0} X, i_0^*(z))$.

Let us denote as $H(\mathfrak{a})$ the 0-th homology of the total complex $Tot(\mathfrak{a})$ of \mathfrak{a} . In other words, $H(\mathfrak{a})$ is the cokernel $\text{Coker}(a_{1,0} \oplus a_{0,1} \xrightarrow{d_{1,0} \oplus d_{0,1}} a_{0,0})$.

If A^* satisfies *(CONST)*, then the restriction to the generic point defines the surjection $A^*(X) \twoheadrightarrow A \rightarrow 0$ which has a canonical splitting given by the pull-back $p_X^* : A = A^*(\text{Spec}(k)) \rightarrow A^*(X)$. Let us denote as $\overline{A}^*(X)$ the kernel $\text{Ker}(A^*(X) \rightarrow A)$. Thus, $A^*(X) = \overline{A}^*(X) \oplus A$ canonically.

The push-forwards define natural map $a_{0,0} \rightarrow \overline{A}_*(X)$, and it follows from Proposition 7.15 that it descends to a map $\theta_{\mathfrak{a}} : H(\mathfrak{a}) \rightarrow \overline{A}_*(X)$. By the *(EXCI)* and the resolution of singularities (Theorem 8.2), this map is surjective.

Definition 4.1 *Let A^* be generalized oriented cohomology theory in the sense of Definition 2.1 satisfying (CONST). We say that A^* is "of rational type" if the map $\theta_{\mathfrak{a}} : H(\mathfrak{a}) \rightarrow \overline{A}_*(X)$ is an isomorphism.*

Remark 4.2 *Not all constant theories are of rational type. For example, CH_{alg} - the Chow groups modulo algebraic equivalence is not such. Indeed, in this case, for a curve C of genus $g > 0$, the map $\theta_{\mathfrak{a}}$ becomes $\overline{\text{CH}}(C) \rightarrow \overline{\text{CH}}_{\text{alg}}(C)$, and has large kernel.*

But below we will see that all the "standard theories" are rational.

Proposition 4.3 *Algebraic cobordism Ω^* of M.Levine-F.Morel is a theory of rational type.*

Proof: The main tool here is the comparison result of M.Levine stating that Ω^n is a "pure part" $MGL^{2n,n}$ of MGL . By the proof of Theorem 3.1 from [7], we have an exact sequence:

$$\mathbb{Z}[k(X)^\times] \otimes \mathbb{L}_{*'} \xrightarrow{\text{div}^\Omega} \Omega_*^{(1)}(X) \rightarrow \Omega_*(X) \rightarrow \mathbb{L} \rightarrow 0.$$

Here $\Omega_*^{(1)}(X) = \lim_{\substack{\rightarrow \\ W}} \Omega_*(W)$, where the limit is taken over all closed subvarieties different from X . The

map div^Ω is \mathbb{L} -linear and is defined on $\mathbb{Z}[k(X)^\times]$ as follows: for a rational function $f \in k(X)^\times$ one resolves the indeterminacy of f using Theorem 8.3 by the permitted blow up $\pi : \tilde{X} \rightarrow X$ making $\tilde{f} : \tilde{X} \rightarrow \mathbb{P}^1$ a morphism, and $\tilde{X}_0 = \tilde{f}^{-1}(0) \rightarrow \tilde{X}$, $\tilde{X}_\infty = \tilde{f}^{-1}(\infty) \rightarrow \tilde{X}$ - the divisors with normal crossing. Then $\text{div}^\Omega(f) = \pi_*([\tilde{X}_0] - [\tilde{X}_\infty])$, where $[\tilde{X}_0], [\tilde{X}_\infty] \in (\Omega^{(1)})^1(\tilde{X})$ are classes of divisors with normal crossing - see Definition 7.12.

For any theory A^* we can define the following bi-complex* $\mathfrak{a}' = \mathfrak{a}'(A^*)$:

- $a'_{0,0} := \bigoplus_{V \rightarrow X} A_*(V)$, where the sum is taken over all projective maps $V \rightarrow X$, such that V is smooth and $\dim(\text{image}(V)) < \dim(X)$;
- $a'_{1,0} := \bigoplus_{V' \rightarrow V \rightarrow X} A_*(V')$, where the sum is taken over all projective composable maps $V' \xrightarrow{\pi} V \rightarrow X$ such that V and V' are smooth, and $\dim(\text{image}(V)) < \dim(X)$;
- $a'_{0,1} := \bigoplus_{W \rightarrow X \times \mathbb{P}^1} A_{*+1}(W)$, where the sum is taken over all projective maps $w : W \rightarrow X \times \mathbb{P}^1$ with W smooth, $\dim(\text{image}(W)) \leq \dim(X)$, and $W_0 = w^{-1}(X \times \{0\}) \xrightarrow{i_0} W$, $W_1 = w^{-1}(X \times \{1\}) \xrightarrow{i_1} W$ divisors with normal crossing.

and the differentials and $H(\mathfrak{a}')$ are defined as for \mathfrak{a} .

Now, let $A^* = \Omega^*$, and $\mathfrak{a}' = \mathfrak{a}'(\Omega^*)$. The push-forwards provide the natural map $a'_{0,0} \rightarrow \Omega_*^{(1)}$, which clearly descends to the map $\alpha' : \text{Coker}(d_{1,0}^{\mathfrak{a}'} \rightarrow \Omega_*^{(1)}(X)$.

Lemma 4.4 *The map $\alpha' : \text{Coker}(d_{1,0}^{\mathfrak{a}'} \rightarrow \Omega_*^{(1)}(X)$ is an isomorphism.*

Proof: Since $\text{Coker}(d_{1,0}^{\mathfrak{a}'}) = \lim_{\substack{\rightarrow \\ V \rightarrow X}} \Omega_*(V)$, where the limit is taken over all projective maps $v : V \rightarrow X$ with the image - closed subscheme of positive codimension in X , and $\Omega_*(Z) = \lim_{\substack{\rightarrow \\ V \rightarrow Z}} \Omega_*(V)$, we get:

$$\Omega_*^{(1)}(X) = \lim_{\substack{\rightarrow \\ Z}} \Omega_*(Z) = \lim_{\substack{\rightarrow \\ V \rightarrow X}} \Omega_*(V) = \text{Coker}(d_{1,0}^{\mathfrak{a}'}).$$

□

From here it is easy to see that $H(\mathfrak{a}') \rightarrow \overline{\Omega}_*(X)$ is an isomorphism, but we will compare \mathfrak{a}' and \mathfrak{a} first.

We have a natural map of bi-complexes $\mathfrak{a} \rightarrow \mathfrak{a}'$ which gives us the map $\alpha : \text{Coker}(d_{1,0}^{\mathfrak{a}} \rightarrow \text{Coker}(d_{1,0}^{\mathfrak{a}'})$, and $\hat{\alpha} : \text{Coker}(\hat{d}_{0,1}^{\mathfrak{a}} \rightarrow \text{Coker}(\hat{d}_{0,1}^{\mathfrak{a}'}))$, where $\hat{d}_{0,1}^{\mathfrak{a}}$ is the map $a_{0,1} \rightarrow \text{Coker}(d_{1,0}^{\mathfrak{a}})$ (and similar for \mathfrak{a}').

Lemma 4.5 *For any theory A^* (in the sense of Definition 2.1), the map*

$$\alpha : \text{Coker}(d_{1,0}^{\mathfrak{a}} \rightarrow \text{Coker}(d_{1,0}^{\mathfrak{a}'})$$

is an isomorphism.

Proof: (surjectivity) Let $Z = \text{image}(V) \xrightarrow{i} X$, and $\pi_z : \tilde{Z} \rightarrow Z$ - Hironaka's resolution of singularities (Theorem 8.2). Then by another result of Hironaka (Theorem 8.3) we can resolve the indeterminacy of the rational map $V \dashrightarrow \tilde{Z}$ by blowing V at smooth centers producing the following commutative diagram:

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{\pi_v} & V \\ \tilde{v} \downarrow & & \downarrow v \\ \tilde{Z} & \xrightarrow{\pi_z} & Z. \end{array} \quad (1)$$

Since $(\pi_v)_* : A_*(\tilde{V}) \rightarrow A_*(V)$ is surjective, we have: $(i \circ v, x) = (i \circ v \circ \pi_v, y) = (i \circ \pi_z, \tilde{f}_*(y))$, for some $y \in A_*(\tilde{V})$, and $\dim(\tilde{Z}) < \dim(X)$.

(injectivity) To prove injectivity, consider the filtration \mathfrak{a}'^r on \mathfrak{a}' defined by additional condition: $\dim(V) < r, \dim(V') < r, \dim(W) < r + 1$. Then $\mathfrak{a}'^\infty = \mathfrak{a}'$, and $\mathfrak{a}'^{\dim(X)} = \mathfrak{a}$. Let us show that for $r > \dim(X)$, the inclusion $\mathfrak{a}'^{r-1} \subset \mathfrak{a}'^r$ induces an isomorphism $\text{Coker}(d_{1,0}^{\mathfrak{a}'^{r-1}}) = \text{Coker}(d_{1,0}^{\mathfrak{a}'^r})$. We need to prove injectivity only. Let $x = \sum_v(v, x_v) \in \mathfrak{a}'_{0,0}^{r-1}$ is such that there exists $y = \sum_{f: V' \rightarrow V}(f, y_f) \in \mathfrak{a}'_{1,0}^r$ such that $d_{1,0}(y) = x$. Let us list all $v : V \rightarrow X$ and $v' : V' \rightarrow X$ participating in y , and for each such V resolve the singularities of $Z = \text{image}(V)$ by Hironaka, and resolving the indeterminacy of the rational map $V \dashrightarrow \tilde{Z}$ using Theorem 8.3 complete the commutative diagram (1), where π_v is the permitted blow up with smooth centers.

Lemma 4.6 Any map $f : V' \rightarrow V$ and the choice of diagrams of type (1) for V and V' can be completed to diagrams:

$$\begin{array}{ccccc} & & \tilde{V}' & \xrightarrow{\pi_{v'}} & V' \\ & \nearrow \tilde{f}_2 & \downarrow \tilde{v}' & \searrow f & \downarrow v' \\ \tilde{V} & \xrightarrow{\pi_v} & V & & \\ \downarrow \tilde{v} & \nearrow h_f & \downarrow T_f & \searrow \tilde{f} & \\ \tilde{Z} & \xrightarrow{\pi_z} & Z & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{Z}' & \xrightarrow{j'} & \tilde{X}' \\ \tilde{f}_2 \uparrow & & \uparrow \varepsilon_2 \\ T_f \xrightarrow{e} & Y_f & \\ \tilde{f}_1 \downarrow & & \downarrow \varepsilon_1 \\ \tilde{Z} & \xrightarrow{j} & \tilde{X}. \end{array} \quad (2)$$

where $\tilde{f}_2, \varepsilon_1, \varepsilon_2$ are projective birational, \tilde{f}_1 is the composition of a projective birational and a closed embedding, j, j', e are regular embeddings, and $\tilde{X}, \tilde{X}', Y_f$ are projective birational over X .

Proof: Since all the fibers of π_v are rational varieties, there exists a rational map $g : \tilde{V}' \dashrightarrow \tilde{V}$ lifting f . Let R_f be the the closure of the image of

$$\tilde{V}' \xrightarrow{(\tilde{v} \circ g, v' \circ \pi_{v'})} \tilde{Z} \times_Z Z'.$$

Then $R_f \xrightarrow{g_1} \tilde{Z} \xrightarrow{j} \tilde{X}$, where j is a regular embedding, and g_1 is a closed one. Also, the rational map $\tilde{X} \dashrightarrow \tilde{X}'$ is defined in the generic point of R_f , and we have commutative diagram:

$$\begin{array}{ccc} \tilde{Z}' & \dashleftarrow & R_f \\ j' \downarrow & & \downarrow j \circ g_1 \\ \tilde{X}' & \dashleftarrow & \tilde{X}. \end{array}$$

Then, by Theorems 8.2, 8.3, there exists a permitted blow up $\tilde{Y} \rightarrow \tilde{X}$, which resolves indeterminacy of this map, and makes the proper preimage T_f of R_f smooth. We get the commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xleftarrow{\tilde{f}_1} T_f \xrightarrow{\tilde{f}_2} & \tilde{Z}' \\ \pi_z \downarrow & & \downarrow \pi_{z'} \\ Z & \xleftarrow{\bar{f}} & Z' \end{array}$$

Then, by Theorem 8.3, there exists a blow up with smooth centers $\tilde{f}_2 : W_f \rightarrow \tilde{V}'$ which fits into the diagram (2). \square

Let $f : V' \rightarrow V$ be a map appearing in y . Apply Lemma 4.6 to f and diagrams of type (1) chosen for $V \rightarrow X$ and $V' \rightarrow X$. We get diagram (2).

Notice, that the front and the rear faces of our diagram do not depend on f , but only on v and v' . Let $y = \sum_f (f, y_f)$. The maps π_V and $\pi_{V'}$ are birational, so $(\pi_V)_*(1)$ and $(\pi_{V'})_*(1)$ are invertible. Consider $b_f := (\pi_{V'})^* \left(\frac{y_f}{(\pi_{V'})_*(1)} \right) \in A_*(\tilde{V}')$, and $c_f := (\pi_V)^* \left(\frac{f_*(y_f)}{(\pi_V)_*(1)} \right) \in A_*(\tilde{V})$. Since \tilde{f}_2 is birational, we can find $a_f \in A_*(W_f)$ such that $(\tilde{f}_2)_*(a_f) = b_f$. Define $d_f = (h_f)_*(a_f)$.

The map π_v is the permitted blow up, so by Proposition 7.6(1), we have an exact sequence:

$$0 \leftarrow A_*(V) \xleftarrow{(\pi_v)^*} A_*(\tilde{V}) \leftarrow \text{Ker}(A_*(E) \xrightarrow{p_*} A_*(R)),$$

where $E = \coprod_i E_i$ - the disjoint union of the components of the exceptional divisor of the blow up, and $R = \coprod_i R_i$ is the disjoint union of the respective smooth centers. Since $(\pi_v)_*(c_f - (\tilde{f}_1)_*(a_f)) = 0$, there exists $e_f \in A_*(E)$ such that $p_*(e_f) = 0$ and $\tilde{q}_*(e_f) = c_f - (\tilde{f}_1)_*(a_f)$, where p and \tilde{q} fit into the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{q}} & \tilde{V} \\ p \downarrow & & \downarrow \pi_v \\ R & \xrightarrow{q} & V. \end{array}$$

Consider

$$t_f := (\pi_v, c_f) - (\tilde{v}, c_f) + (\tilde{f}_1, d_f) - (\tilde{f}_2, d_f) + (\tilde{v}', b_f) - (\pi_{V'}, b_f) + (\tilde{v} \circ \tilde{q}, e_f) - (p, e_f), \quad \text{and} \quad t = \sum_f t_f.$$

Since $d_{1,0}(y) \in a'_{0,0}{}^{r-1}$, we have that for every $v : V \rightarrow X$ of dimension r , we have: $\sum_{\tilde{f} \rightarrow V} f_*(y_f) - \sum_{V \xrightarrow{\tilde{f}}} y_f = 0$.

Then

$$\sum_f ((\pi_v, c_f) - (\tilde{v}, c_f) + (\tilde{v}', b_f) - (\pi_{V'}, b_f)) = \sum_v \left(\left(\pi_v, \frac{\sum_{\tilde{f} \rightarrow V} f_*(y_f) - \sum_{V \xrightarrow{\tilde{f}}} y_f}{(\pi_v)_*(1)} \right) - \left(\tilde{v}, \frac{\sum_{\tilde{f} \rightarrow V} f_*(y_f) - \sum_{V \xrightarrow{\tilde{f}}} y_f}{(\pi_v)_*(1)} \right) \right) = 0.$$

And the remaining terms of t_f are of dimension $\leq r - 1$ (notice, that \bar{f} is a closed embedding, and $\dim(T_f) \leq \dim(Z) < \dim(X)$). Hence, $t \in a'_{1,0}{}^{r-1}$. On the other hand, $d_{1,0}(t) = d_{1,0}(y)$ (instead of the path $V' - V$ we are moving: $V' - \tilde{V}' - \tilde{Z}' - T_f - \tilde{Z} - \tilde{V} - V$, and correct the discrepancy on \tilde{Z} with the e_f -terms). Hence, the map $\text{Coker}(d_{1,0}^{a'{}^{r-1}}) \rightarrow \text{Coker}(d_{1,0}^{a'{}^r})$ is injective, which implies that $\text{Coker}(d_{1,0}^a) = \text{Coker}(d_{1,0}^{a'})$. \square

Return to the case $A^* = \Omega^*$. We obtain the commutative diagram with exact columns:

$$\begin{array}{ccccc}
a_{0,1} & \longrightarrow & a'_{0,1} & & \mathbb{Z}[k(X)^\times] \otimes \mathbb{L} \\
\hat{d}_{0,1}^a \downarrow & & \hat{d}_{0,1}^{a'} \downarrow & & \downarrow \text{div} \\
\text{Coker}(d_{1,0}^a) & \xrightarrow{\alpha} & \text{Coker}(d_{1,0}^{a'}) & \xrightarrow{\alpha'} & \Omega_*^{(1)}(X) \\
\downarrow & & \downarrow & & \downarrow \\
H(\mathfrak{a}) & \xrightarrow{\hat{\alpha}} & H(\mathfrak{a}') & \xrightarrow{\hat{\alpha}'} & \bar{\Omega}_*(X) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

where α and α' are isomorphisms. It remains to observe that the map div can be factored through $a_{0,1}$ by the very definition. This shows that the maps

$$H(\mathfrak{a}) \xrightarrow{\hat{\alpha}} H(\mathfrak{a}') \xrightarrow{\hat{\alpha}'} \bar{\Omega}_*(X)$$

are isomorphisms. \square

Note, that more generally, we have:

Lemma 4.7 *For any theory A^* (in the sense of Definition 2.1) the map*

$$H(\mathfrak{a}) \xrightarrow{\hat{\alpha}} H(\mathfrak{a}')$$

is an isomorphism.

Proof: We already know by Lemma 4.5 that $\text{Coker}(d_{1,0}^a) \xrightarrow{\alpha} \text{Coker}(d_{1,0}^{a'})$ is an isomorphism. So, we need to prove only the injectivity of $\hat{\alpha}$.

Let $(U, x) \in a'_{0,1}$, where $i \circ u : U \rightarrow X \times \mathbb{P}^1$, s.t. the preimages U_0 and U_1 are divisors with normal crossing on U , and $T = \text{image}(U) \xrightarrow{i} X \times \mathbb{P}^1$ has dimension $\leq \dim(X)$. Let $\tilde{T} \xrightarrow{\pi_T} T$ - Hironaka's resolution of singularities (see Theorem 8.2) such that the preimages \tilde{T}_0 and \tilde{T}_1 of 0 and 1 on \tilde{T} are divisors with normal crossing, which using the resolution of the indeterminacy of the rational map $(\pi_T)^{-1} \circ u$ (see Theorem 8.3) can be embedded into the commutative diagram:

$$\begin{array}{ccc}
\tilde{U} & \xrightarrow{\tilde{u}} & \tilde{T} \\
\pi_U \downarrow & & \downarrow \pi_T \\
U & \xrightarrow{u} & T,
\end{array}$$

where the preimages \tilde{U}_0 and \tilde{U}_1 of 0 and 1 on \tilde{U} are also divisors with strict normal crossing. For $l = 0, 1$ we have the diagram with Cartesian squares:

$$\begin{array}{ccccc}
\tilde{T} & \xleftarrow{\tilde{u}} & \tilde{U} & \xrightarrow{\pi_u} & U \\
i_{l,\tilde{T}} \uparrow & & i_{l,\tilde{U}} \uparrow & & \uparrow i_{l,U} \\
\tilde{T}_l & \xleftarrow{\tilde{u}_l} & \tilde{U}_l & \xrightarrow{\pi_{u,l}} & U_l
\end{array}$$

Since π_u is birational, the map $(\pi_u)_*$ is surjective. Hence, there exists $y \in A_*(\tilde{U})$ such that $(\pi_U)_*(y) = x$. By the "Multiple points excess intersection formula" (Proposition 7.21), $d_{0,1}(U, x) = d_{0,1}(\tilde{U}, y) = d_{0,1}(\tilde{T}, \tilde{u}_*(y))$, and $(\tilde{T}, \tilde{u}_*(y)) \in a_{0,1}$, since $\dim(\tilde{T}) \leq \dim(X)$. Thus, $\hat{\alpha}$ is an isomorphism. \square

Proposition 4.3 permits to describe all theories of rational type. It appears that these are exactly the *free* theories in the sense of M.Levine and F.Morel - see [8, Remark 2.4.14].

Proposition 4.8 *Let A^* be the theory (in the sense of Definition 2.1) satisfying (CONST). Then*

$$A^* \text{ is of rational type} \Leftrightarrow A^* = \Omega^* \otimes_{\mathbb{L}} A.$$

In particular, there is a 1-to-1 correspondence between rational theories A^ and formal group laws (A, F_{A^*}) .*

Proof: Since the tensor product functor is exact from the right, any theory of the type $\Omega^* \otimes_{\mathbb{L}} A$ will be of rational type by the very definition, since Ω^* is (Proposition 4.3).

Conversely, suppose A^* is of rational type. By the result of M.Levine-F.Morel, Ω^* is the universal theory, so we get a canonical morphism $\Omega^* \rightarrow A^*$ which extends to $\Omega^* \otimes_{\mathbb{L}} A \xrightarrow{g} A^*$ (since A acts on A^*). This morphism is an isomorphism for X of dimension zero. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} (a_{1,0} \oplus a_{0,1})(\Omega^* \otimes_{\mathbb{L}} A) & \longrightarrow & a_{0,0}(\Omega^* \otimes_{\mathbb{L}} A) & \longrightarrow & \overline{(\Omega^* \otimes_{\mathbb{L}} A)}(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \bar{g} & & \\ (a_{1,0} \oplus a_{0,1})(A^*) & \longrightarrow & a_{0,0}(A^*) & \longrightarrow & \overline{A^*}(X) & \longrightarrow & 0 \end{array} \quad (3)$$

Since $a_{0,0}(B^*) = \bigoplus_{\substack{V \rightarrow X \\ \dim(V) < \dim(X)}} B_*(V)$, and g is an isomorphism on points, by induction on the dimension

of X we see that g is surjective for all X (recalling that $B^*(Y) = \overline{B^*}(Y) \oplus B$). Again using the above diagram, we prove by induction on the dimension that g is an isomorphism for all X . \square

Remark: Although, in the end, our *theories of rational type* appear to be the same as *free* theories of M.Levine-F.Morel, there are several justifications for this alternative approach. The main ones being that it gives the definition which is internal, and permits to argue inductively on dimension, which will be crucial in our dealing with unstable operations below. Also, this definition is rather flexible and permits generalizations - see Subsection 4.4.

By results of M.Levine and F.Morel ([8, Theorems 1.2.18, 1.2.19]), the (usual) Chow groups CH^* and K_0 are *free* theories, and hence, the theories of rational type. Other "standard (pure) theories" such as BP^* and higher Morava K-theories $K(n)$ are, by definition, obtained from algebraic cobordism by change of coefficients, and so are of rational type as well. In other words, everything which was "pulled" from topology gives pure theories of rational type. This is related to the fact that in topology all spectra are made of "cellular" spaces, so their algebraic counterparts have motives constructed out of Tate-motives.

Let A^* be any theory satisfying (CONST). Let us denote as $(A^{(0)})^*$ the theory $\Omega^* \otimes_{\mathbb{L}} A$. Then we have the canonical map $(A^{(0)})^* \xrightarrow{g} A^*$.

Proposition 4.9 (1) $g : (A^{(0)})^* \twoheadrightarrow A^*$ is surjective;

- (2) (cf. [8, Remark 2.4.14]) Any morphism $A^* \xrightarrow{f} B^*$ of theories extends to a canonical commutative diagram:

$$\begin{array}{ccc} (A^{(0)})^* & \xrightarrow{f^{(0)}} & (B^{(0)})^* \\ g_A \downarrow & & \downarrow g_B \\ A^* & \xrightarrow{f} & B^* \end{array}$$

In particular, the assignment $A^* \mapsto (A^{(0)})^*$ is a functor on the category of theories. The assignment: $f \mapsto f^{(0)}$ is injective.

- (3) ([8, Remark 2.4.14]) Morphisms $A^* \rightarrow B^*$ between theories of rational type are in 1-to-1 correspondence with the morphisms of \mathbb{L} -algebras $A \rightarrow B$.

Proof: (1) Since $a_{0,0}(A^*) \rightarrow \overline{A}^*(X)$ is surjective for any theory satisfying (EXCI), the surjectivity of g follows by induction on the dimension of X .

(2) Restriction to $\text{Spec}(k)$ gives the morphism of \mathbb{L} -algebras $A \rightarrow B$, and hence a morphism of theories $(A^{(0)})^* \xrightarrow{f^{(0)}} (B^{(0)})^*$. The fact that the respective diagram is commutative follows from universality of Ω^* , as the composition $\Omega^* \rightarrow (A^{(0)})^* \rightarrow A^* \xrightarrow{f} B^*$ can also be decomposed as $\Omega^* \rightarrow (B^{(0)})^* \rightarrow B^*$, and $(B^{(0)})^* = (A^{(0)})^* \otimes_A B$. The injectivity $f \mapsto f^{(0)}$ follows from (1).

(3) Follows immediately from universality of Ω^* . \square

Let us now give some alternative descriptions of *rational theories*, which will permit us to work with unstable operations effectively.

4.2 The bi-complex* \mathfrak{b} .

The bi-complex* $\mathfrak{a}(A^*)$ describes the theory A^* of X in terms of A^* of smaller-dimensional varieties and push-forward maps. But, to be able to work with cohomological operations we will need to find the presentation in terms of pull-backs. This will be done in two steps. First, we will enhance the classes (U, x) from \mathfrak{a} by requiring U to sit in some \tilde{X} projective bi-rational over X , but still keeping push-forwards - this is done in the current Section. And then, in the next one, we will switch from push-forwards to pull-backs by roughly "going down and up" via the projection $\tilde{X} \rightarrow X$.

Let us move to the first step. Consider the bi-complex* $\mathfrak{b} = \mathfrak{b}(A^*)$:

$$\cdot \quad b_{0,0} := \bigoplus_{\substack{\tilde{X} \rightarrow X \\ V \subset \tilde{X}}} A_*(V), \text{ where the sum is taken over all projective birational maps } \tilde{X} \rightarrow X \text{ which are}$$

isomorphisms on $\tilde{X} \setminus V$, with \tilde{X} - smooth, and $V \rightarrow \tilde{X}$ - divisor with strict normal crossing;

$$\cdot \quad b_{1,0} := \left(\bigoplus_{\substack{\tilde{X} \xrightarrow{\pi} \tilde{X} \rightarrow X \\ V' \subset \tilde{X}, V \subset \tilde{X}}} A_*(V') \right) \oplus \left(\bigoplus_{\substack{\tilde{X} \rightarrow X \\ V_1 \subset V_2 \subset \tilde{X}}} A_*(V_1) \right), \text{ where the first sum is taken over all composable}$$

projective birational maps $\tilde{X} \xrightarrow{\pi} \tilde{X} \rightarrow X$ with smooth \tilde{X}, \tilde{X} , which are isomorphisms on $\tilde{X} \setminus V'$ and $\tilde{X} \setminus V$; $V' \rightarrow \tilde{X}$ and $V \rightarrow \tilde{X}$ are divisors with strict normal crossing; π - permitted blow up with respect to V ; $V' = \pi^{-1}(V)$; and the second sum is taken over all projective birational maps $\tilde{X} \rightarrow X$, which are isomorphisms over $\tilde{X} \setminus V_1$, and $V_1 \subset V_2$ are divisors with strict normal crossing on \tilde{X} .

$$\cdot \quad b_{0,1} := \bigoplus_{\substack{\widetilde{X \times \mathbb{P}^1} \rightarrow X \times \mathbb{P}^1 \\ W \subset \widetilde{X \times \mathbb{P}^1}}} A_{*+1}(W), \text{ where the sum is taken over all permitted (with respect to } X \times 0,$$

$X \times 1)$ blow ups $\widetilde{X \times \mathbb{P}^1} \rightarrow X \times \mathbb{P}^1$, which are isomorphisms on $\widetilde{X \times \mathbb{P}^1} \setminus W$, where W is a divisor with strict normal crossing, having no components over 0 and 1, and for each component S of W : $S_0 = s^{-1}(X \times \{0\}) \xrightarrow{i_0} S$, $S_1 = s^{-1}(X \times \{1\}) \xrightarrow{i_1} S$ are divisors with strict normal crossing.

and the differentials are defined as follows:

$$\cdot \quad d_{1,0} \left(\begin{pmatrix} V' \rightarrow \widetilde{\widetilde{X}} \\ \pi' \downarrow \quad \downarrow \pi \\ V \rightarrow \widetilde{\widetilde{X}} \end{pmatrix}, y \right) = (V \rightarrow \widetilde{\widetilde{X}}, \pi'_*(y)) - (V' \rightarrow \widetilde{\widetilde{X}}, y) \text{ and}$$

$$d_{1,0}(V_1 \xrightarrow{j} V_2 \rightarrow \widetilde{\widetilde{X}}, y) = (V_2 \rightarrow \widetilde{\widetilde{X}}, j_*(y)) - (V_1 \rightarrow \widetilde{\widetilde{X}}, y).$$

$$\cdot \quad d_{0,1}(W \rightarrow \widetilde{X \times \mathbb{P}^1}, \sum_S z_s) = (W_1 \rightarrow \widetilde{X}_1, \sum_S i_1^*(z_s)) - (W_0 \rightarrow \widetilde{X}_0, \sum_S i_0^*(z_s)),$$

where $\widetilde{X}_0, \widetilde{X}_1$ are preimages (= proper transforms) of $X \times 0$ and $X \times 1$, and $W_l = W \cap \widetilde{X}_l$.

Let us denote as $H(\mathbf{b})$ the 0-th homology of the total complex $Tot(\mathbf{b})$ of \mathbf{b} . We have natural maps:

$$\beta : \text{Coker}(d_{1,0}^{\mathbf{b}}) \rightarrow \text{Coker}(d_{1,0}^{\mathbf{a}}), \quad \text{and} \quad \hat{\beta} : H(\mathbf{b}) \rightarrow H(\mathbf{a}).$$

Let X be smooth quasi-projective variety, and $Z \subset X$ - a closed subscheme. Let $\pi : \widetilde{X} \rightarrow X$ be a permitted blow up at centers over Z such that $W = \pi^{-1}(Z)$ is a strict normal crossing divisor. By the result of Hironaka - see Theorem 8.4, such π always exists. Then we say that the element $(\pi, x) \in b_{0,0}$, where $x \in A_*(W)$ "is defined over Z ". Let us denote as im_Z the subgroup of all "defined over Z elements" in $b_{0,0}$ /(the first half of $d_{1,0}^{\mathbf{b}}$). We have a well-defined map: $\pi_* : im_Z \rightarrow A_*(Z)$.

Proposition 4.10

- (1) *The subgroup im_Z does not depend on the choice of resolution.*
- (2) $im_Z = im_{Z_{red}}$.

Proof: Let $\pi_1 : X_1 \rightarrow X$ and $\pi_2 : X_2 \rightarrow X$ be two resolutions as above (with the same $X \setminus Z$), and $x_1 \in A_*(W_1)$ be some element. We have a birational map $\pi_2^{-1} \circ \pi_1 : X_1 \dashrightarrow X_2$ which is an isomorphism outside W_1 and W_2 . By the Weak Factorization Theorem (see Theorem 8.6(6)), there exists the diagram

$$\begin{array}{ccccccc} & Y_1 & & Y_3 & & Y_{n-2} & & Y_n \\ & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\ X_1 & & Y_2 & & Y_4 & \dots & Y_{n-3} & & Y_{n-1} & & X_2 \end{array}$$

----- π -----

of smooth projective varieties over X where each blow up is permitted w.r. to all components of the preimage of Z . Since the respective push-forwards are surjective, and each projection $Y_i \rightarrow X$ is an isomorphism outside Z , we can "transfer" x_1 from X_1 to X_2 using elements from the first part of the $image(d_{1,0}^{\mathbf{b}})$. \square

Let $Z \subset Z'$ be an embedding of closed subschemes. Then we have the map $\lim_{\substack{\rightarrow \\ V \rightarrow Z}} A_*(V) \rightarrow \lim_{\substack{\rightarrow \\ V \rightarrow Z'}} A_*(V')$, which gives the push-forward: $A_*(Z) \rightarrow A_*(Z')$. Consider the limit $\lim_{\substack{\rightarrow \\ Z \subset X}} A_*(Z)$ over all proper closed subvarieties of X . In analogy with Lemma 4.4 we have:

$$\lim_{\substack{\rightarrow \\ Z \subset X}} A_*(Z) = \text{Coker}(d_{1,0}^a).$$

Lemma 4.11 *Let $T \subset X$ be closed subscheme, and $Y \subset X$ be a divisor smooth outside T . Let $(\pi, x) \in b_{0,0}$ be an element defined over T . Then there exists an element $(\rho, x') \in b_{0,0}$ defined over $T \cup Y$ such that*

$$(\pi, x) \equiv (\rho, x') \text{ mod } \text{image}(d_{1,0}^b), \text{ and } \pi_*(x)|_{T \cup Y} = \rho_*(x').$$

Proof: By permitted blow up $\rho : \tilde{X} \rightarrow X$ with centers over T we can make the preimages $W_1 \xrightarrow{i} W_2$ of $T \subset T \cup Y$ simultaneously divisors with normal crossing. By Proposition 4.10 we can assume that x is defined for this resolution. And using the second part of the $\text{image}(d_{1,0}^b)$, we get $x' := i_*(x)$ whose image in $A_*(T \cup Y)$ is $\pi_*(x)|_{T \cup Y}$ (notice that ρ is an isomorphism outside T). \square

Lemma 4.12 *Let $Z \subset X$ be a proper closed subvariety. Then there exist divisors $Y_i, i = 1, \dots, m$ such that $Z \subset \cup_i Y_i$, and Y_j is smooth outside $\cup_{i=1}^{j-1} Y_i$.*

Proof: Use Noetherian induction. The base ($Z = \emptyset$) is trivial. Suppose, we know the statement for all proper closed subvarieties of Z . By Proposition 7.10, there exist a divisor Y of X which contains Z , and is smooth outside Z , and in the generic points of the components of Z . Thus, the locus of singular points S of Y is a proper subvariety of Z . By induction, there exist divisors $Y_i, i = 1, \dots, m$ such that $S \subset \cup_i Y_i$, and Y_j is smooth outside $\cup_{i=1}^{j-1} Y_i$. Then the divisors $Y_i, i = 1, \dots, m$ together with Y satisfy the conditions for Z . \square

Proposition 4.13 *Let $Z \subset X$ be closed subvariety, $\pi : \tilde{X} \rightarrow X$ be resolution as above (with $W = \pi^{-1}(Z)$), and $x \in A_*(W)$ be such element that $\pi_*(x) = 0 \in \lim_{Z' \subset X} A_*(Z')$. Then there exists $Z' \supset Z$ with the resolution $\pi' : \tilde{X}' \rightarrow X$ (with $W' = \pi'^{-1}(Z')$) and $x' \in A_*(W')$ such that $(\pi', x') \equiv (\pi, x) \text{ mod } \text{image}(d_{1,0}^b)$, and $\pi'_*(x') = 0 \in A_*(Z')$.*

Proof: Let $Z'' \supset Z$ be such closed subvariety that $\pi_*(x)|_{Z''} = 0$. By Lemma 4.12, starting from any closed subvariety Z and adding divisors as in Lemma 4.11 one can get a closed subvariety Z' containing any given Z'' . Since $\pi_*(x)|_{Z''} = 0$, so is $\pi_*(x)|_{Z'}$, which is equal to the push-forward of x' , by Lemma 4.11. \square

Proposition 4.14 *Let $u \in b_{0,0}$ be an element whose image in $\lim_{Z' \subset X} A_*(Z')$ is zero. Then $u \in \text{image}(d_{1,0}^b)$.*

Proof: Let Z_1, Z_2 be two closed subvarieties. Then it follows from Lemma 4.12 that by adding divisors as in Lemma 4.11 one can obtain from Z_1 and Z_2 the same closed subvariety Z (equal to the union of the respective Y_i 's for both sets). Then it follows from Lemma 4.11 that we can assume that $u = (\pi, x)$ is defined over one set Z . From Proposition 4.13 we can assume that $\pi_*(x) = 0 \in A_*(Z)$. Let us prove by Noetherian induction (or induction on the dimension of Z , if you want) that then $u \in \text{image}(d_{1,0}^b)$. If Z is empty, there is nothing to prove. Suppose, we know the statement for proper closed subvarieties of Z . By Proposition 7.10, there exists a divisor Y containing Z and smooth outside some proper closed subvariety S of Z . By Lemma 4.11, we can assume that u is defined over Y . Now, we can choose a

different resolution $\rho : \tilde{X} \rightarrow X$ of Y which involves only blow ups with centers over S and resolves S simultaneously. By Proposition 4.10, we can assume that u is defined for this resolution. Let \tilde{Y} be the proper preimage of Y , E_i be all the components of the exceptional divisor of ρ , and R_i be the respective smooth centers. We have: $u = (\rho, x)$, where $\rho_*(x) = 0$. By Lemma 7.9, we have an exact sequence:

$$0 \leftarrow A_*(Y) \leftarrow \left(A_*(\tilde{Y}) \oplus A_*(S) \right) \leftarrow \oplus_i A_*(E_i \cap \tilde{Y}).$$

From the diagram:

$$\begin{array}{ccccc} \tilde{Y} & \xleftarrow{\quad} & \cup_i (E_i \cap \tilde{Y}) & & \\ \rho' \downarrow & & \downarrow & \searrow & \\ Y & \xleftarrow{\quad j \quad} & S & \xleftarrow{\quad} & \cup_i E_i \end{array}$$

we see that x can be represented by an element from $A_*(\cup_i E_i)$ which projects to zero in $A_*(S)$. Thus, modulo the $\text{image}(d_{1,0}^b)$, u is equal to an element defined over S which vanishes in $A_*(S)$. By induction, $u \in \text{image}(d_{1,0}^b)$. \square

Corollary 4.15 *Let A^* be any theory in the sense of Definition 2.1. Then the natural map $\text{Coker}(d_{1,0}^b) \rightarrow \text{Coker}(d_{1,0}^a)$ is an isomorphism.*

Proof: The surjectivity follows from the existence of resolution making given closed subvariety Z a divisor with normal crossing by permitted blow up over Z (Theorem 8.4), and the fact that all fibers of such a resolution are rational varieties (so, admit sections). The injectivity follows from Proposition 4.14. \square

Proposition 4.16 *For any theory A^* (in the sense of Definition 2.1) satisfying (CONST), the natural map*

$$\hat{\beta} : H(\mathfrak{b}) \rightarrow H(\mathfrak{a})$$

is an isomorphism.

Proof: By Corollary 4.15 and Lemma 4.5, the map $\beta : \text{Coker}(d_{1,0}^b) \rightarrow \text{Coker}(d_{1,0}^a)$ is an isomorphism. So we need to prove the injectivity of $\hat{\beta}$ only.

Let us introduce:

$$\cdot \quad b'_{0,1} := \bigoplus_{\substack{\widetilde{X \times \mathbb{P}^1} \rightarrow X \times \mathbb{P}^1 \\ W \subset \widetilde{X \times \mathbb{P}^1}}} A_{*+1}(W), \text{ where the sum is taken over all projective birational maps } \widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho}$$

$X \times \mathbb{P}^1$, which are isomorphisms on $\widetilde{X \times \mathbb{P}^1} \setminus W$, where W is a divisor with strict normal crossing which is in good position with the preimages of $X \times \{0\}$ and $X \times \{1\}$ (that is, the union of these divisors is a divisor with strict normal crossing) which are also divisors with strict normal crossing.

In particular, for each component S of W , either S is over 0 or 1, or: $S_0 = s^{-1}(X \times \{0\}) \xrightarrow{i_0} S$, $S_1 = s^{-1}(X \times \{1\}) \xrightarrow{i_1} S$ are divisors with strict normal crossing.

Define $d_{0,1}^{b'} : b'_{0,1} \rightarrow a_{0,0}$ by the formula:

$$\cdot \quad d_{0,1}(W \rightarrow \widetilde{X \times \mathbb{P}^1}, \sum_S z_s) = (W_1 \rightarrow \tilde{X}_1 \xrightarrow{\rho_1} X, \sum_{S \rightarrow \mathbb{P}^1 - \text{dom.}} (\rho_1)_* i_1^*(z_s)) - (W_0 \rightarrow \tilde{X}_0 \xrightarrow{\rho_0} X, \sum_{S \rightarrow \mathbb{P}^1 - \text{dom.}} (\rho_0)_* i_0^*(z_s)),$$

where \tilde{X}_0, \tilde{X}_1 are proper preimages of $X \times 0$ and $X \times 1$, and $W_l = W \cap \tilde{X}_l$.

The basic difference between $b'_{0,1}$ and $b_{0,1}$ is that in the latter we do not permit blow ups with centers located over 0 and 1, but we do not require components of W to be in normal crossing with the preimages of 0 and 1.

Let us denote the images of the respective maps:

$$a_{0,1} \xrightarrow{\hat{d}_{0,1}^a} \text{Coker}(d_{1,0}^a); \quad b'_{0,1} \xrightarrow{\hat{d}_{0,1}^{b'}} \text{Coker}(d_{1,0}^a); \quad b_{0,1} \xrightarrow{\hat{d}_{0,1}^b} \text{Coker}(d_{1,0}^a)$$

as $\text{im}(\hat{d}_{0,1}^a)$, $\text{im}(\hat{d}_{0,1}^{b'})$, $\text{im}(\hat{d}_{0,1}^b)$, respectively

Let $Z \subset X \times \mathbb{P}^1$ be some closed subscheme, and $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho} X \times \mathbb{P}^1$ be some projective birational map which is an isomorphism outside Z , and such that the preimages W, X_0, X_1 of Z and $X \times \{0\}$, $X \times \{1\}$ are divisors with strict normal crossing, and so is their union. By Theorem 8.4 and Proposition 8.5, such a resolution always exists. Let us denote as im_Z the $\text{image}(A_{*+1}(W) \xrightarrow{\hat{d}_{0,1}^{b'}} \text{Coker}(d_{1,0}^a))$.

Lemma 4.17

- (1) The subgroup im_Z does not depend on the choice of the resolution $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho} X \times \mathbb{P}^1$.
- (2) $\text{im}_Z = \text{im}_{Z_{\text{red}}}$.

Proof: Let $Q_1 \xrightarrow{\rho_1} X \times \mathbb{P}^1$ and $Q_2 \xrightarrow{\rho_2} X \times \mathbb{P}^1$ be two such resolutions with $\rho_1^{-1}(Z_1) = W_1$, $\rho_2^{-1}(Z_2) = W_2$ with $(Z_1)_{\text{red}} = (Z_2)_{\text{red}}$. In particular, Q_1 and Q_2 are isomorphic outside the preimages of $Z_i \cup X \times \{0\} \cup X \times \{1\}$. By the Weak Factorization Theorem (Theorem 8.6), there exists a diagram:

$$\begin{array}{ccccccc} & & Y_1 & & Y_3 & & \\ & \swarrow & & \searrow & \swarrow & \searrow & \\ Q_1 & & Y_2 & & Y_4 & \dots & Y_{n-3} & & Y_{n-2} & & Y_n & & Q_2 \\ & \nwarrow & & \swarrow & \nwarrow & \swarrow & & \nwarrow & & \swarrow & & \nwarrow & \end{array}$$

of smooth projective varieties over $X \times \mathbb{P}^1$ where each blow up is permitted w.r. to all components of the preimage of $Z_i \cup X \times \{0\} \cup X \times \{1\}$. The respective push-forwards are surjective and commute with $d_{0,1}$ by Proposition 7.21 (although, Y_i 's are of slightly more general type than $\widetilde{X \times \mathbb{P}^1}$ from $b'_{0,1}$, we can consider the respective elements in $a_{0,1}$ and the differential $d_{0,1}^a$ on them). Notice also, that components dominant over \mathbb{P}^1 are mapped to dominant ones. Hence, im_Z will be the same. \square

Lemma 4.18 Let $(U \xrightarrow{f} X \times \mathbb{P}^1, x) \in a_{0,1}$, and $T = f(U) \subset X \times \mathbb{P}^1$. Then

$$\hat{d}_{0,1}^a(U, x) \in \text{im}_T.$$

Proof: Let $(U, x) \in a_{0,1}$, where $U \xrightarrow{f} X \times \mathbb{P}^1$, is such that the preimages U_0 and U_1 of $X \times \{0\}$ and $X \times \{1\}$ are divisors with strict normal crossing on U , and $T = f(U) \xrightarrow{i} X \times \mathbb{P}^1$ has dimension $\leq \dim(X)$. We can assume U irreducible. Let $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\pi} X \times \mathbb{P}^1$ be the resolution of T making it's preimage \tilde{T} as well as its union with the preimages of $X \times \{0\}$ and $X \times \{1\}$ - divisors with strict normal crossing (Theorem 8.4, Proposition 8.5). Since the fibers of π are rational varieties (and so, admit sections), we get a rational map $U \dashrightarrow S$ to some component $S \xrightarrow{j_S} \tilde{T}$. Notice, that S is dominant over \mathbb{P}^1 . Resolving the indeterminacy of this map and making the preimages \tilde{U}_0 and \tilde{U}_1 of 0 and 1 on \tilde{U} divisors with strict normal crossing (Theorems 8.3 and 8.4), we get the commutative diagram:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{u}} & S \\ \pi_U \downarrow & & \downarrow \pi_S \\ U & \xrightarrow{u} & T. \end{array}$$

For $l = 0, 1$ we have the diagram with Cartesian squares:

$$\begin{array}{ccccc} S & \xleftarrow{\tilde{u}} & \tilde{U} & \xrightarrow{\pi_u} & U \\ i_{l,S} \uparrow & & i_{l,\tilde{U}} \uparrow & & \uparrow i_{l,U} \\ S_l & \xleftarrow{\tilde{u}_l} & \tilde{U}_l & \xrightarrow{\pi_{u,l}} & U_l \end{array}$$

Since π_u is birational, the map $(\pi_u)_*$ is surjective. Hence, there exists $y \in A_*(\tilde{U})$ such that $(\pi_u)_*(y) = x$. By the "Multiple points excess intersection formula" (Proposition 7.21),

$$\hat{d}_{0,1}^a(U, x) = \hat{d}_{0,1}^a(\tilde{U}, y) = \hat{d}_{0,1}^a(S, \tilde{u}_*(y)) = \hat{d}_{0,1}^{b'}(\tilde{T} \subset \widetilde{X \times \mathbb{P}^1}, (j_S)_* \circ \tilde{u}_*(y)).$$

Thus, $\hat{d}_{0,1}^a(U, x) \in im_T$. □

Lemma 4.19 *For any theory A^* (in the sense of Definition 2.1),*

$$im(\hat{d}_{0,1}^a) = im(\hat{d}_{0,1}^{b'}).$$

Proof: Clearly, $im(\hat{d}_{0,1}^{b'}) \subset im(\hat{d}_{0,1}^a)$, since the diagram:

$$\begin{array}{ccc} b'_{0,1} & \xrightarrow{d_{0,1}^{b'}} & a_{0,0} \\ \text{forg.} \downarrow & & \parallel \\ a_{0,1} & \xrightarrow{d_{0,1}^a} & a_{0,0}, \end{array}$$

is commutative. The other inclusion follows from Lemma 4.18. □

Lemma 4.20 *Let $Z \subset X \times \mathbb{P}^1$ be a closed subvariety, and $Y \subset X \times \mathbb{P}^1$ be a divisor such that $Y \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with strict normal crossing outside Z . Then*

$$im_Z \subset im_{Z \cup Y}.$$

Proof: By Proposition 8.5 and Theorem 8.4, we can find a blow up $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho} X \times \mathbb{P}^1$ permitted w.r.to $X \times \{0\}$ and $X \times \{1\}$ with centers over Z , such that the preimages of $Z, Z \cup Y, X \times \{0\}$, and $X \times \{1\}$ are strict normal crossing divisors, and so is their union. By Lemma 4.17, we can use this resolution to define im_Z and $im_{Z \cup Y}$. The inclusion follows. □

Lemma 4.21 *Let $Z \subset X \times \mathbb{P}^1$ be a closed subvariety whose projection to X has dimension $< \dim(X)$ (for example, $\dim(Z) < \dim(X)$). Then*

$$im_Z \subset im(\hat{d}_{0,1}^b).$$

Proof: We have: $\dim(p(Z)) < \dim(X)$, where $p : X \times \mathbb{P}^1 \rightarrow X$ is the projection. By Lemma 4.12, there exist divisors $T_i, i = 1, \dots, m$ on X such that T_i is smooth outside $\cup_{j=1}^{i-1} T_j$, and $\cup_{i=1}^m T_i =: T \supset p(Z)$. Then $Y_i = T_i \times \mathbb{P}^1, i = 1, \dots, m$ will have the property: $Y_i \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with normal crossing outside $\cup_{j=1}^{i-1} Y_j$, and $\cup_{i=1}^m Y_i \supset Z$. By Lemma 4.20, $im_Z \subset im_Y$, where $Y = \cup_{i=1}^m Y_i = T \times \mathbb{P}^1$. Now we can make the preimage of $Y \cup X \times \{0\} \cup X \times \{1\}$ a divisor with normal crossing by blowing $X \times \mathbb{P}^1$ at smooth centers of the form $B \times \mathbb{P}^1$ located over $T \times \mathbb{P}^1$. Thus, the preimage \tilde{Y} of Y will not have components over 0 or 1, and the preimages of $X \times \{0\}$ and $X \times \{1\}$ will be birational to X (and consist each of one component only). Hence, the elements of im_Y will be coming from $b_{0,1}$. □

Lemma 4.22 *Let $Z \subset X \times \mathbb{P}^1$ be a closed subvariety with the resolution $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho} X \times \mathbb{P}^1$, $W = \rho^{-1}(Z)$ - the divisor with normal crossing, and $u = (\rho, x) \in b'_{0,1}$ be such element that $\rho_*(x) = 0 \in A_{*+1}(Z)$. Then*

$$\hat{d}_{0,1}^{b'}(u) \in \text{im}(\hat{d}_{0,1}^b).$$

Proof: Using Lemma 4.20 and Lemma 4.12 (as in the proof of the Lemma 4.21), we can make Z a divisor. Let $T \subset Z$ be a closed subvariety outside which $Z \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with normal crossing. Then $\dim(T) < \dim(Z) = \dim(X)$. Let $\widetilde{X \times \mathbb{P}^1} \rightarrow X \times \mathbb{P}^1$ be the resolution making the preimages of $Z \cup X \times \{0\} \cup X \times \{1\}$ and of $T \cup X \times \{0\} \cup X \times \{1\}$ - divisors with normal crossing by blowing at smooth centers permitted w.r.to $X \times \{0\}$ and $X \times \{1\}$ and located over T . By Lemma 4.17, we can assume that u is defined for this resolution. Let \tilde{Z} be the proper preimage of Z , and E_i be all the components of the exceptional divisor of ρ . We have: $u = (\rho, x)$, where $\rho_*(x) = 0$. By Lemma 7.9, we have an exact sequence:

$$0 \leftarrow A_*(Z) \leftarrow (A_*(\tilde{Z}) \oplus A_*(T)) \leftarrow \oplus_i A_*(E_i).$$

From the diagram:

$$\begin{array}{ccccc} \tilde{Z} & \xleftarrow{\cup_i (E_i \cap \tilde{Z})} & & & \\ \rho' \downarrow & & \downarrow & \searrow & \\ Z & \xleftarrow{j} & T & \xleftarrow{\cup_i E_i} & \end{array}$$

we see that x can be represented by an element from $A_*(\cup_i E_i)$ which projects to zero in $A_*(T)$. Thus, we can assume that u is defined over T . Since $\dim(T) < \dim(X)$, by Lemma 4.21, $\hat{d}_{0,1}^{b'}(u) \in \text{im}(\hat{d}_{0,1}^b)$. \square

Lemma 4.23 *Let $Z \subset X \times \mathbb{P}^1$ be a proper closed subvariety. Then there exist divisors Y_i , $i = 1, \dots, m$ such that $Z \subset \cup_i Y_i$, and $Y_j \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with strict normal crossing outside $\cup_{i=1}^{j-1} Y_i$.*

Proof: Let $Z' \subset Z$ be the closure of $Z \setminus (X \times \{0\} \cup X \times \{1\})$, and $S' = (Z' \cap (X \times \{0\} \cup X \times \{1\})) \cup \text{Sing}(Z) \cup Z_{<\dim(X)}$ (where the latter is the union of the smaller dimensional (non-divisorial) components). Let $S := p(S')_{\text{red}}$, where $p : X \times \mathbb{P}^1 \rightarrow X$ is the projection. Then $\dim(S) < \dim(X)$. By Lemma 4.12, there are divisors T_i , $i = 1, \dots, m$ on X such that T_i is smooth outside $\cup_{j=1}^{i-1} T_j$, and $\cup_{i=1}^m T_i \supset S$. Then, clearly, the divisors $Y_i := T_i \times \mathbb{P}^1$ on $X \times \mathbb{P}^1$ will have the property that $Y_i \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with normal crossing outside $\cup_{j=1}^{i-1} Y_j$, and on $(X \times \mathbb{P}^1) \setminus \cup_{i=1}^m Y_i$ our closed subvariety Z is a smooth divisor not meeting $(X \times \{0\} \cup X \times \{1\})$. It remains to add the divisorial part of Z to Y_i , $i = 1, \dots, m$. \square

Lemma 4.24 (1) *Let $Z = Z_1 \cup Z_2$, where Z_1, Z_2 be closed subvarieties of $X \times \mathbb{P}^1$. Then*

$$\text{im}_Z \equiv \text{im}_{Z_1} + \text{im}_{Z_2} \text{ mod } \text{im}(\hat{d}_{0,1}^b).$$

(2) *Let $u = (\rho, x) \in b'_{0,1}$ be an element defined over a closed subvariety Z such that the restriction of $\rho_*(x)$ to the generic points of all the components of Z of dimension $= \dim(X)$ is zero. Then $\hat{d}_{0,1}^{b'}(u) \in \text{im}(\hat{d}_{0,1}^b)$.*

Proof: By Lemma 4.23, we can find divisors Y_i , $i = 1, \dots, m$ such that $Y_i \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with strict normal crossing outside $\cup_{j=1}^{i-1} Y_j$, and $Y = \cup_{i=1}^m Y_i \supset Z$.

(1) We have: the map $A_{*+1}(Z_1) \oplus A_{*+1}(Z_2) \twoheadrightarrow A_{*+1}(Z)$ is surjective, and so are the maps $\rho_* : A_{*+1}(W) \rightarrow A_{*+1}(Z)$, $(\rho_1)_* : A_{*+1}(W_1) \rightarrow A_{*+1}(Z_1)$, and $(\rho_2)_* : A_{*+1}(W_2) \rightarrow A_{*+1}(Z_2)$ (the fibers are rational varieties). But if we have elements u, u_1, u_2 defined over Z, Z_1 , and Z_2 , respectively, such that $\rho_*(u) = (\rho_1)_*(u_1)|_Z + (\rho_2)_*(u_2)|_Z$, then by the proof of Lemma 4.20, there are elements u', u'_1, u'_2 defined over Y

which represent the same elements in $\text{Coker}(d_{1,0}^a)$ as u, u_1, u_2 and such that their images in $A_{*+1}(Y)$ are $\rho_*(u)|_Y, (\rho_1)_*(u_1)|_Y, (\rho_2)_*(u_2)|_Y$, respectively. It remains to apply Lemma 4.22.

(2) By condition, $\rho_*(x)$ is defined over some subvariety $S \subset Z$ of dimension $< \dim(X)$. Using the above Y again together with Lemmas 4.20, 4.21, and 4.22, we get what we need. \square

Lemma 4.25 *Let $Z \subset X \times \mathbb{P}^1$ be a divisor of degree 1 over X . Then:*

$$im_Z \in im(\hat{d}_{0,1}^b).$$

Proof: It will be convenient to change a parametrization of \mathbb{P}^1 , so that the points 0 and ∞ be marked (instead of 0 and 1). By our condition, Z is a closure of the graph of some meromorphic function $f \in k(X)^\times$. Let D_0 and D_∞ - be the divisors of zeroes and poles of f (we can clearly assume that f is not constant). By blowing $\tilde{X} \xrightarrow{\pi} X$ at smooth centers located over $D_0 \cup D_\infty$ we can make the proper preimages \tilde{D}_0 and \tilde{D}_∞ (we take the proper preimages of components keeping the multiplicities), the special divisor E of π , and $\tilde{D}_0 \cup E, \tilde{D}_\infty \cup E$ to be divisors with strict normal crossing, and the function $f : \tilde{X} \rightarrow \mathbb{P}^1$ be everywhere defined (by Theorem 8.3 and Proposition 8.5). Let $\widetilde{X \times \mathbb{P}^1} = \tilde{X} \times \mathbb{P}^1$. Thus, it is obtained from $X \times \mathbb{P}^1$ by blow ups constant along \mathbb{P}^1 . We have the regular embedding: $\tilde{X} \cong \Gamma_f \rightarrow \widetilde{X \times \mathbb{P}^1}$, and $\Gamma_f \cap (\tilde{X} \times 0) = \tilde{D}_0 + E_0$, and $\Gamma_f \cap (\tilde{X} \times \infty) = \tilde{D}_\infty + E_\infty$, where E_0 and E_∞ are some $\mathbb{Z}_{\geq 0}$ -linear combinations of the components of E . Thus, these are divisors with strict normal crossing on Γ_f . Let $F = E \times \mathbb{P}^1$ be the special divisor on $\widetilde{X \times \mathbb{P}^1}$. By our condition, the divisor $W = \Gamma_f \cup F$ has strict normal crossing. By blowing $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho} \widetilde{X \times \mathbb{P}^1}$ further at smooth centers located over $W \cap (\tilde{X} \times \{0\})$ and $W \cap (\tilde{X} \times \{\infty\})$ we can make the preimage \widetilde{W} of W , as well as it's union with the preimages of $(\tilde{X} \times \{0\})$ and $(\tilde{X} \times \{\infty\})$ to be divisors with strict normal crossing. Let $\tilde{u} \in b'_{0,1}$ be an arbitrary element defined over Z . By Lemma 4.17(1), we can assume that it has the form: $\tilde{u} = (\widetilde{W \rightarrow X \times \mathbb{P}^1}, \tilde{x})$, for some $\tilde{x} \in A_{*+1}(\widetilde{W})$. Let $u = (W \rightarrow X \times \mathbb{P}^1, \rho_*(\tilde{x})) \in b_{0,1}$. By Proposition 7.21, the push-forward ρ_* commutes with the pull-backs i_0^* and i_∞^* . Hence, $\hat{d}_{0,1}^{b'}(\tilde{u}) = \hat{d}_{0,1}^b(u) \in \text{Coker}(d_{1,0}^a)$. Hence, $im_Z \subset im(\hat{d}_{0,1}^b)$. \square

Let $Y \subset X \times \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible divisor. We will say that Y is "in general position", if it's intersections with all the faces of $\mathbb{P}^1 \times \mathbb{P}^1$ have the right dimension, and Y is smooth in the generic points of all the components of the intersection with the large faces. That is, $Y_{s=l}$ and $Y_{t=l}$ are divisors in $X \times \mathbb{P}^1$, for $l = 0, 1$, and $Y_{s=l, t=m}$ are divisors in X , for $l, m = 0, 1$. We call a divisor $Z \subset X \times \mathbb{P}^1$ "constant" if it has the form $Q \times \mathbb{P}^1$.

Lemma 4.26 *Let A^* be any theory in the sense of Definition 2.1 satisfying (CONST). Let $Y \subset X \times \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible divisor in general position, and such that $Y_{s=1} = Z \cup (\text{constant div.})$, where Z is irreducible, and the divisors Y and $(s = 1)$ are transversal in the generic point of Z . Then:*

$$im_Z \subset im_{Y_{s=0}} + im_{Y_{t=0}} + im_{Y_{t=1}} + im(\hat{d}_{0,1}^b).$$

Proof: Let $\widetilde{X \times \mathbb{P}^1 \times \mathbb{P}^1} \xrightarrow{\pi} X \times \mathbb{P}^1 \times \mathbb{P}^1$ be the permitted blow up with centers over Y making the proper preimage \tilde{Y} of Y a smooth divisor, the proper preimage of $(\cup_{l=0}^1 X \times \mathbb{P}^1 \times \{l\}) \cup (\cup_{m=0}^1 X \times \{m\} \times \mathbb{P}^1)$ - a divisor with strict normal crossing, and it's intersection with \tilde{Y} - a divisor with strict normal crossing on \tilde{Y} . Then $\pi^{-1}(X \times \mathbb{P}^1 \times \{l\})$ consist of the proper preimage $\tilde{T}_{s=l}$ of $X \times \mathbb{P}^1 \times \{l\}$ and, possibly, of some components $\tilde{E}_{s=l}$ located over some closed subvarieties $R_{s=l}$ of $Y \cap X \times \mathbb{P}^1 \times \{l\}$. Moreover, since this intersection is smooth in the generic point, the mentioned subvarieties will be of codimension ≥ 2

on $X \times \mathbb{P}^1 \times \{l\}$ (that is, of dimension $< \dim(X)$). And the same happens to $\pi^{-1}(X \times \{m\} \times \mathbb{P}^1)$. Let

$$\begin{array}{ccc} X \times \{m\} \times \{l\} & \xrightarrow{j'_{t=m}} & X \times \mathbb{P}^1 \times \{l\} \\ j'_{s=l} \downarrow & & \downarrow j_{s=l} \\ X \times \{m\} \times \mathbb{P}^1 & \xrightarrow{j_{t=m}} & X \times \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \tilde{Y}_{t=m,s=l} & \xrightarrow{i'_{t=m}} & \tilde{Y}_{s=l} \\ i'_{s=l} \downarrow & & \downarrow i_{s=l} \\ \tilde{Y}_{t=m} & \xrightarrow{i_{t=m}} & \tilde{Y}, \end{array}$$

be the face maps and their restriction on \tilde{Y} . By the functoriality of i^* we have:

$$(i'_{t=m})^* \circ (i_{s=l})^* = (i'_{s=l})^* \circ (i_{t=m})^* : A_{*+2}(\tilde{Y}) \rightarrow A_*(\tilde{Y}_{t=m,s=l}).$$

If $x \in A_{*+2}(\tilde{Y})$, then $(i_{s=l})^*(x)$ will have components on $\tilde{T}_{s=l}$ representing some element of $b'_{0,1}$, and, possibly, some components on $\tilde{E}_{s=l}$. Considering the latter components as elements of $a_{0,1}$, we see from Lemma 4.18 that $\hat{d}_{0,1}^a$ of these in $\text{Coker}(d_{1,0}^a)$ will belong to $im_{R_{s=l}}$, which is a subset of $im(\hat{d}_{0,1}^b)$, by Lemma 4.21, since $\dim(R_{s=l}) < \dim(X)$. Thus, modulo $im(\hat{d}_{0,1}^b)$, we can ignore these components. In the same way, by the same Lemma we can ignore the components located over constant divisors on $X \times \mathbb{P}^1$. Let $\overline{(i_{s=l})^*}(x) \in b'_{0,1}$ denotes the $\tilde{T}_{s=l}$ -part of $(i_{s=l})^*(x)$ with constant components ignored (well-defined modulo elements coming from $\tilde{E}_{s=l}$ and modulo constant components). Similar considerations apply to $(i_{t=m})^*(x)$. We have:

$$\hat{d}_{0,1}^{b'} \circ \overline{(i_{s=1})^*}(x) - \hat{d}_{0,1}^{b'} \circ \overline{(i_{s=0})^*}(x) \equiv \hat{d}_{0,1}^{b'} \circ \overline{(i_{t=1})^*}(x) - \hat{d}_{0,1}^{b'} \circ \overline{(i_{t=0})^*}(x) \mod im(\hat{d}_{0,1}^b)$$

Let $\tilde{i} : \tilde{Z} \rightarrow \tilde{Y}$ be the proper preimage of Z . Then \tilde{Z} is a component of $\tilde{Y} \cap \tilde{T}_{s=1}$, and by our condition (on the restriction of faces to \tilde{Y}), \tilde{Z} is a smooth divisor on \tilde{Y} . Moreover, since the intersection of Y and $X \times \mathbb{P}^1 \times \{1\}$ was transversal in the generic point of Z , we have:

$$\pi^*(\mathcal{O}(s=1))|_{\tilde{Y}} = \mathcal{O}(\tilde{Z}) \otimes \mathcal{O}(C) \otimes \mathcal{O}(\tilde{E}'_{s=1}),$$

where C is some constant divisor, and $\tilde{E}'_{s=1}$ is some $\mathbb{Z}_{\geq 0}$ -linear combination of the components of $\tilde{E}_{s=1}$, which implies that we can set: $\overline{(i_{s=1})^*}(x) = \overline{(i)^*}(x)$. The map $\tilde{Z} \rightarrow Z$ is birational, and the composition

$$A_{*+2}(\tilde{Y}) \xrightarrow{(\tilde{i})^*} A_{*+1}(\tilde{Z}) \rightarrow A_{*'}(k(\tilde{Z}))$$

is surjective, because A^* is constant (this is the only (!) place where we are using the *(CONST)* axiom). Thus, for any $a \in A_{*+1}(k(Z))$ we can find an element $x \in A_{*+2}(\tilde{Y})$ such that $\pi_*((\overline{(i_{s=1})^*}(x))|_{k(Z)}) = a$, while (by the above identity) $\hat{d}_{0,1}^{b'}(\overline{(i_{s=1})^*}(x)) \in im_{Y_{s=0}} + im_{Y_{t=1}} + im_{Y_{t=0}} + im(\hat{d}_{0,1}^b)$. It remains to apply Lemma 4.24(2). \square

Lemma 4.27 *For any theory A^* (in the sense of Definition 2.1) satisfying (CONST),*

$$im(\hat{d}_{0,1}^b) = im(\hat{d}_{0,1}^{b'}).$$

Proof: By Lemma 4.21, it remains to show that $im_Z \subset im(\hat{d}_{0,1}^b)$, for each divisor $Z \subset X \times \mathbb{P}^1$. By Lemmas 4.24 and 4.21 we can assume that Z is irreducible, and has positive degree over X , and by definition of $\hat{d}_{0,1}^{b'}$ we can assume that Z is different from $X \times \{0\}$ and $X \times \{1\}$, and so, meets these divisors in proper codimension. Let $Z^\eta \subset \mathbb{P}_{k(X)}^1$ be the restriction of Z w.r.to the open embedding $\text{Spec}(k(X)) \rightarrow X$. Then Z^η is some closed point of some degree $n > 0$ on the projective line over $k(X)$. Let $\sum_{i=0}^n a_i t_1^i t_0^{n-i}$ with $a_i \in k(X)$ be the homogeneous equation of Z^η . By our condition, $a_0 \neq 0$ and

$\sum_{i=0}^n a_i \neq 0$. Consider $Y^\eta \subset (\mathbb{P}^1 \times \mathbb{P}^1)_{k(X)}$ defined by the equation: $\sum_{i=1}^n a_i t_1^i t_0^{n-i} s_1 + a_0 t_0^n s_0$. Then $Y_{s=1}^\eta = Z^\eta$, $Y_{s=0}^\eta$ is given by the equation $a_0 t_0^n$, $Y_{t=0}^\eta$ - by the equation $a_0 s_0$, and $Y_{t=1}^\eta$ - by the equation $(\sum_{i=1}^n a_i) s_1 + a_0 s_0$. Thus, Y^η is a curve in general position on $(\mathbb{P}^1 \times \mathbb{P}^1)_{k(X)}$, whose restriction to $(s=1)$ is Z^η , whose other three restrictions are points of degree 1 (some with multiplicities), and which does not meet "corners" $(s=l, t=m)$, $l, m = 0, 1$. Moreover, Y^η is smooth at all the points $Y_{s=1}^\eta, Y_{s=0}^\eta, Y_{t=1}^\eta, Y_{t=0}^\eta$, and transversal to $(s=1)$ at Z^η . Denote as Y the closure of Y^η in $X \times \mathbb{P}^1 \times \mathbb{P}^1$. Then, modulo constant divisors, $Y_{s=1} \equiv Z$, while $Y_{s=0}, Y_{t=0}$, and $Y_{t=1}$ are equivalent to some divisors of degree 1 over X (some of these with multiplicities), and Y is transversal to $(s=0)$ at the generic point of Z . It follows from Lemmas 4.26, 4.25, 4.21 and 4.17(2) that $im_Z \subset im(\hat{d}_{0,1}^b)$. \square

Combining Lemmas 4.19 and 4.27 we see that $im(\hat{d}_{0,1}^b) = im(\hat{d}_{0,1}^a) \subset \text{Coker}(d_{1,0}^a)$, which implies that the natural map $\hat{\beta} : H(\mathbf{b}) \rightarrow H(\mathbf{a})$ is an isomorphism. Proposition 4.16 is proven. \square

4.3 The bi-complex* \mathbf{c} .

Now we are ready to construct the description of A^* in terms of pull-backs. In this section we will assume that A_* is a *theory of rational type*. By Proposition 4.8, that means that A_* is *free* in the sense of M.Levine-F.Morel, that is, it can be obtained from Algebraic Cobordism theory by change of coefficients: $A_* = \Omega_* \otimes_{\mathbb{L}} A$. In particular, we can use the tools constructed by M.Levine and F.Morel for Algebraic Cobordism in [8]. Among them we will need the *refined pull-backs* for locally complete intersection morphisms.

Consider the bi-complex* $\mathbf{c} = \mathbf{c}(A^*)$:

$$\cdot \quad c_{0,0} := \bigoplus_{\substack{\tilde{X} \xrightarrow{\rho} X \\ V \subset \tilde{X}}} A_*(V) \cap im(\rho^!), \text{ where the sum is taken over all projective birational maps } \tilde{X} \xrightarrow{\rho} X$$

which are isomorphisms on $\tilde{X} \setminus V$, with \tilde{X} - smooth, $V \rightarrow \tilde{X}$ - strict normal crossing divisor, and $\rho^! : A_*(\rho(V)) \rightarrow A_*(V)$ is the refined pull-back;

$$\cdot \quad c_{1,0} := \left(\bigoplus_{\substack{\tilde{X} \xrightarrow{\pi} \tilde{X} \xrightarrow{\rho} X \\ V' \subset \tilde{X}, V \subset \tilde{X}}} A_*(V) \cap im(\rho^!) \right) \oplus \left(\bigoplus_{\substack{\tilde{X} \xrightarrow{\rho} X \\ V_1 \subset V_2 \subset \tilde{X}}} A_*(V_1) \cap im(\rho^!) \right), \text{ where the first sum is taken}$$

over all composable projective birational maps $\tilde{X} \xrightarrow{\pi} \tilde{X} \xrightarrow{\rho} X$ with smooth \tilde{X}, \tilde{X} , which are isomorphisms on $\tilde{X} \setminus V'$ and $\tilde{X} \setminus V$; $V' \rightarrow \tilde{X}$ and $V \rightarrow \tilde{X}$ are strict normal crossing divisors; π - single permitted blow up with respect to V ; $V' = \pi^{-1}(V)$; $\rho^! : A_*(\rho(V)) \rightarrow A_*(V)$ is the refined pull-back; and the second sum is taken over all projective birational maps $\tilde{X} \xrightarrow{\rho} X$, which are isomorphisms over $\tilde{X} \setminus V_1$, and $V_1 \subset V_2$ are strict normal crossing divisors on \tilde{X} .

$$\cdot \quad c_{0,1} := \bigoplus_{\substack{\widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho} X \times \mathbb{P}^1 \\ W \subset X \times \mathbb{P}^1}} A_{*+1}(W) \cap im(\rho^!), \text{ where the sum is taken over all permitted (with respect to}$$

$X \times 0, X \times 1$) blowings up $\widetilde{X \times \mathbb{P}^1} \rightarrow X \times \mathbb{P}^1$, which are isomorphisms on $\widetilde{X \times \mathbb{P}^1} \setminus W$, where W is a strict normal crossing divisor, having no components over 0 and 1, and for each component S of W : $S_0 = s^{-1}(X \times \{0\}) \xrightarrow{i_0} S$, $S_1 = s^{-1}(X \times \{1\}) \xrightarrow{i_1} S$ are divisors with strict normal crossing, and $\rho^!$ is the refined pull-back.

and the differentials are defined as follows:

$$\begin{aligned}
& \cdot d_{1,0}(\left(\begin{array}{ccc} V' & \xrightarrow{\widetilde{X}} & \widetilde{X} \\ \pi_V \downarrow & & \downarrow \pi \\ V & \xrightarrow{\quad} & \widetilde{X} \end{array} , x \right)) = (V \rightarrow \widetilde{X}, x) - (V' \rightarrow \widetilde{X}, \pi^!(x)) \text{ and} \\
& d_{1,0}(V_1 \xrightarrow{j} V_2 \rightarrow \widetilde{X}, y) = (V_2 \rightarrow \widetilde{X}, j_*(y)) - (V_1 \rightarrow \widetilde{X}, y). \\
& \cdot d_{0,1}(W \rightarrow \widetilde{X \times \mathbb{P}^1}, \sum_S z_s) = (W_1 \rightarrow \widetilde{X}_1, \sum_S i_1^*(z_s)) - (W_0 \rightarrow \widetilde{X}_0, \sum_S i_0^*(z_s)), \\
& \text{where } \widetilde{X}_0, \widetilde{X}_1 \text{ are proper transforms (= preimages) of } X \times 0 \text{ and } X \times 1, \text{ and } W_l = W \cap \widetilde{X}_l.
\end{aligned}$$

The fact that $d_{1,0}$ lands in $c_{0,0}$ follows from functoriality of the refined pull-backs, and from commutativity of them with push-forwards ([8, Theorem 6.6.6(3),(2)(a)]).

Our $W_l = \cup_S S_l$ fits into the cartesian diagram (with $\overline{W} = \rho(W)$, and $\overline{W}_l = \overline{W} \cap X \times \{l\}$):

$$\begin{array}{ccccc}
\overline{W}_l & \xleftarrow{\rho'_l} & W_l & \xrightarrow{j'_l} & W & \xrightarrow{\rho'} & \overline{W} & & \widetilde{X}_l & \xrightarrow{j_l} & \widetilde{X \times \mathbb{P}^1} \\
\overline{w}_l \downarrow & & w_l \downarrow & & w \downarrow & & \overline{w} \downarrow & & \rho_l \downarrow & & \downarrow \rho \\
X \times \{l\} & \xleftarrow{\rho_l} & \widetilde{X}_l & \xrightarrow{j_l} & \widetilde{X \times \mathbb{P}^1} & \xrightarrow{\rho} & X \times \mathbb{P}^1, & & X \times \{l\} & \xrightarrow{k_l} & X \times \mathbb{P}^1,
\end{array}$$

And since the blow up $\widetilde{X \times \mathbb{P}^1} \rightarrow X \times \mathbb{P}^1$ has no centers located over 0 and 1, it is transversal to the embeddings $X \times \{l\} \xrightarrow{k_l} X \times \mathbb{P}^1$, $l = 0, 1$. This implies that $(k_l)^! = (j_l)^! : A_{*+1}(W) \rightarrow A_*(W_l)$ (see [8, Lemma 6.6.2]). Now, the fact that $d_{0,1}$ lands in $c_{0,0}$ follows from the equality (for $l = 0, 1$): $\sum_S i_l^*(z_s) = (k_l)^!(\sum_S z_s)$, and from functoriality of the refined pull-backs:

$$(j_l)^! \circ \rho^! = (\rho \circ j_l)^! = (\rho_l)^! \circ (k_l)^!.$$

Consider the map:

$$\rho^! \circ \rho_* : \mathfrak{b} \rightarrow \mathfrak{c}.$$

The fact that it commutes with the first half of $d_{1,0}$ follows from the identity $(\rho \circ \pi)^! \circ (\rho \circ \pi)_* = \pi^! \circ (\rho^! \circ \rho_*) \circ \pi_*$ (which uses the functoriality of push-forwards and refined pull-backs - see [8, Theorem 6.6.6(3)].) The commutativity with the second half of $d_{1,0}$ follows from the fact that the push-forwards are functorial, and that the refined pull-backs commute with push-forwards - see [8, Theorem 6.6.6(2)(a)]. The commutativity with $d_{0,1}$ follows from the equality: $(k_l)^! = (j_l)^! : A_{*+1}(W) \rightarrow A_*(W_l)$, and the identity:

$$j_l^! \circ \rho^! \circ \rho'_* = \rho_l^! \circ k_l^! \circ \rho'_* = \rho_l^! \circ \rho'_{l*} \circ k_l^! = \rho_l^! \circ \rho'_{l*} \circ j_l^!,$$

which again uses [8, Theorem 6.6.6(3),(2)(a)].

Let us denote as $H(\mathfrak{c})$ the 0-th homology of the total complex $Tot(\mathfrak{c})$ of \mathfrak{c} . We have natural maps:

$$\gamma : \text{Coker}(d_{1,0}^b) \rightarrow \text{Coker}(d_{1,0}^c), \quad \text{and} \quad \hat{\gamma} : H(\mathfrak{b}) \rightarrow H(\mathfrak{c}).$$

Since all the fibers of ρ are rational varieties, and $V = \rho^{-1}(\rho(V))$, the map $\rho_* : A_*(V) \rightarrow A_*(\rho(V))$ is surjective, (and similar for W). Hence, the map $\rho^! \circ \rho_* : \mathfrak{b} \rightarrow \mathfrak{c}$ is surjective.

Since A^* is a *theory of rational type*, by Definition 4.1 and Proposition 4.16, we have an identification:

$$\overline{A}^* = H(\mathfrak{b}),$$

which gives the map

$$\varphi : \overline{A}^* \rightarrow H(\mathfrak{c}).$$

On the other hand, we have the map

$$\psi = \frac{\rho_*}{\rho_*(1)} : H(\mathfrak{c}) \rightarrow \overline{A}^*.$$

The fact that ψ is well defined follows from commutativity of push-forwards with the refined pull-backs, functoriality of push-forwards, and projection formula. From the same projection formula we see that the composition $\psi \circ \varphi : \overline{A}^* \rightarrow \overline{A}^*$ is the identity. Thus, we have proven:

Theorem 4.28 *Let A^* be a theory of rational type. Then we have the natural identification:*

$$\overline{A}^* = H(\mathfrak{c}).$$

4.4 Theories of higher types

The Definition 4.1 is sufficiently flexible. In the sense, that one can modify the term $a_{0,1}$ somewhat. In the current form, it imposes *rational equivalence*.

Consider the bi-complex $\mathfrak{a}^{(1)} = \mathfrak{a}^{(1)}(A^*)$, where $a_{0,0}^{(1)} = a_{0,0}$, $a_{1,0}^{(1)} = a_{1,0}$, while $a_{0,1}^{(1)} := \bigoplus_{W \rightarrow X \times C} A_{*+1}(W)$, where the sum is taken over all projective maps $w : W \rightarrow X \times C$ with W smooth, C - smooth projective curve with two fixed points $p_0 \xrightarrow{j_0} C \xleftarrow{j_1} p_1$ of the same degree on it, $\dim(W) \leq \dim(X)$, and $W_0 = w^{-1}(X \times p_0) \xrightarrow{i_0} W$, $W_1 = w^{-1}(X \times p_1) \xrightarrow{i_1} W$ are divisors with strict normal crossing, where the differential $d_{0,1}$ is defined as before.

Define $H(\mathfrak{a}^{(1)})$ as the zero-th homology of $Tot(\mathfrak{a}^{(1)})$. We have a natural surjection $H(\mathfrak{a}) \twoheadrightarrow H(\mathfrak{a}^{(1)})$.

Definition 4.29 *Let A^* be a theory (in the sense of Definition 2.1) satisfying (CONST). We call A^* a "theory of algebraic type", if the natural homomorphism $\theta_{\mathfrak{a}} : H(\mathfrak{a}) \rightarrow \overline{A}^*$ descends to an isomorphism $\theta_{\mathfrak{a}^{(1)}} : H(\mathfrak{a}^{(1)}) \rightarrow \overline{A}^*$.*

Example of such a theory is provided by CH_{alg} - the Chow groups modulo algebraic equivalence.

What is important here is that imposing relations similar to that of $a_{0,1}^?$ produces again a theory in the sense of Definition 2.1. So, one can produce a huge number of "theories", which demonstrates that the general object satisfying this Definition is not particularly good, and additional restrictions are needed if one wants to prove any result of interest.

It is natural to consider the *rational type* as *type 0*, and *algebraic type* as *type 1*. It is tempting to extend this to arbitrary n , producing a tower:

$$(A^{(0)})^* \twoheadrightarrow (A^{(1)})^* \twoheadrightarrow \dots \twoheadrightarrow (A^{(n)})^* \twoheadrightarrow \dots,$$

where $(A^\infty)^* = \text{image}((A^{(0)})^* \rightarrow A_{top}^*)$, and $A^{(n)}$ -equivalence on m -motives, $m \leq n$ would coincide with the topological equivalence. In particular, I would expect the "proper" theories of type n to be again in 1-to-1 correspondence with the formal group laws $\mathbb{L} \rightarrow A$ (and so produced by change of coefficients from the *Algebraic Cobordism of type n*). We will not pursue this direction in the current paper.

5 From products of projective spaces to Sm_k

In this section, A^* is a *theory of rational type*, and B^* is any theory in the sense of Definition 2.1. Our aim here is to prove the main result of the article:

Theorem 5.1 *Let A^* be a theory of rational type, and B^* be any theory in the sense of Definition 2.1. Fix $n, m \in \mathbb{Z}$. Then any additive transformation*

$$A^n((\mathbb{P}^\infty)^{\times l}) \xrightarrow{G} B^m((\mathbb{P}^\infty)^{\times l}), \text{ for } l \in \mathbb{Z}_{\geq 0}$$

commuting with the pull-backs for:

- (i) *the action of \mathfrak{S}_l ;*
- (ii) *the partial diagonals;*
- (iii) *the partial Segre embeddings*

extends to a unique additive operation $A^n \xrightarrow{G} B^m$ on \mathbf{Sm}_k .

Remark 5.2 *The condition on A^* is necessary. For example, the identity transformation $\mathrm{CH}_{\mathrm{alg}}|_{(\mathbb{P}^\infty)^{\times l}} \xrightarrow{id} \mathrm{CH}_{\mathrm{rat}}|_{(\mathbb{P}^\infty)^{\times l}}$ can not be extended to a morphism of theories.*

The transformation $A^n \xrightarrow{G} B^m$ on $(\mathbb{P}^\infty)^{\times l}$ commuting with the partial diagonals is completely described by its action on $\alpha \cdot (\prod_{i=1}^l z_i^A)$, where $z_i^A = c_1^A(\mathcal{O}(1)_i)$ and $\alpha \in A^{n-l}$. Let $G(\alpha \cdot (\prod_{i=1}^l z_i^A)) = G_l(\alpha)(z_1^B, \dots, z_l^B) \in B[[z_1^B, \dots, z_l^B]]_{(m)}$. Conditions (i) and (iii) impose certain restrictions on these. Starting with the power series $G_l(\alpha)(z_1, \dots, z_l)$ we will extend G to $X \times (\mathbb{P}^\infty)^{\times l}$, for arbitrary variety X by induction on the dimension of X .

Definition 5.3 *Let X be smooth quasi-projective variety. Denote as $\mathbf{G}(X) = \{G_l, l \in \mathbb{N}\}$ the following data:*

$$G_l \in \mathrm{Hom}_{\mathbb{Z}\text{-lin}}(A^{n-l}(X), B^*(X)[[z_1, \dots, z_l]]_{(m)}) \text{ satisfying:}$$

- (a_i) *G_l is symmetric with respect to \mathfrak{S}_l ;*
- (a_{ii}) *$G_l(\alpha) = \prod_{i=1}^l z_i \cdot F_l(\alpha)$, for some $F_l(\alpha) \in B^*(X)[[z_1, \dots, z_l]]_{(m-l)}$.*
- (a_{iii}) *$G_l(\alpha)(x +_B y, z_2, \dots, z_l) = \sum_{i,j \geq 1} G_{i+j+l-1}(\alpha \cdot a_{i,j}^A)(x^{\times i}, y^{\times j}, z_2, \dots, z_l)$,
where $a_{i,j}^A$ and $a_{i,j}^B$ are the coefficients of the formal group laws of A^* and B^* .*

Let $\chi_A(x) = (-_A x) = \sum_{i \geq 0} e_i^A \cdot x^{i+1}$, $x -_A y = \sum_{i,j} b_{i,j}^A x^i y^j$, and similar for B . Then it follows from (a_{iii}) and (a_{ii}) that:

$$\begin{aligned} G_l(\alpha)(-_B x, z_2, \dots, z_l) &= \sum_{i \geq 0} G_{l+i}(\alpha \cdot e_i^A)(x^{\times i+1}, z_2, \dots, z_l), \quad \text{and} \\ G_l(\alpha)(x -_B y, z_2, \dots, z_l) &= \sum_{i,j} G_{i+j+l-1}(\alpha \cdot b_{i,j}^A)(x^{\times i}, y^{\times j}, z_2, \dots, z_l). \end{aligned}$$

If \mathcal{V} is some vector bundle with B -roots $\lambda_1^B, \dots, \lambda_r^B$, then it follows from (a_i) that $F_{l+r}(\alpha)(\lambda_1^B, \dots, \lambda_r^B, z_1, \dots, z_l)$ is a function of $c_1^B(\mathcal{V}), \dots, c_r^B(\mathcal{V})$, and so, it does not depend on the choice of roots.

Definition 5.4 *Let d be some natural number. We say that $\mathbf{G}(d)$ is defined, if, for all X smooth quasi-projective of dimension $\leq d$, $\mathbf{G}(X)$ is defined, and these satisfy:*

- (b_i) *For any $f : X \rightarrow Y$ with $\dim(X), \dim(Y) \leq d$, and any $\alpha \in A^{n-l}(Y)$,*

$$G_l(f_A^*(\alpha)) = f_B^* G_l(\alpha).$$

(b_{ii}) For any regular embedding $j : X \rightarrow Y$ of codimension r with normal bundle \mathcal{N}_j with B -roots μ_1^B, \dots, μ_r^B , for any $\alpha \in A^{n-l-r}(X)$, one has:

$$F_l(j_*(\alpha))(z_1, \dots, z_l) = j_*(F_{l+r}(\alpha)(\mu_1^B, \dots, \mu_r^B, z_1, \dots, z_l)).$$

The condition (b_{ii}) can be rewritten as:

$$G_l(j_*(\alpha))(z_1, \dots, z_l) = j_* \operatorname{Res}_{t=0} \frac{G_{l+r}(\alpha)(t +_B \mu_1^B, \dots, t +_B \mu_r^B, z_1, \dots, z_l) \cdot \omega_t^B}{(t +_B \mu_1^B) \cdot \dots \cdot (t +_B \mu_r^B) \cdot t}$$

In such a situation we have the following specialization result.

Lemma 5.5 *Let $\mathbf{G}(d)$ be defined, X be smooth quasi-projective variety of dimension $\leq d$, and \mathcal{L} be a linear bundle on X with $\lambda^A = c_1^A(\mathcal{L})$, $\lambda^B = c_1^B(\mathcal{L})$. Then, for any $\alpha \in A^{n-l-1}(X)$,*

$$G_l(\alpha \cdot \lambda^A)(z_1, \dots, z_l) = G_{l+1}(\alpha)(\lambda^B, z_1, \dots, z_l). \quad (4)$$

Proof: 1) Let \mathcal{L} be very ample. Then $\lambda^A = j_*(1)$, where $Y \xrightarrow{j} X$ is a smooth divisor, and:

$$\begin{aligned} G_l(\alpha \cdot \lambda^A)(z_1, \dots, z_l) &= F_l(j_* j^*(\alpha))(z_1, \dots, z_l) \cdot \prod_{i=1}^l z_i = \\ j_* F_l(j^*(\alpha))(\lambda^B, z_1, \dots, z_l) \cdot \prod_{i=1}^l z_i &= j_* j^* F_l(\alpha)(\lambda^B, z_1, \dots, z_l) \cdot \prod_{i=1}^l z_i = \\ F_l(\alpha)(\lambda^B, z_1, \dots, z_l) \cdot \lambda^B \cdot \prod_{i=1}^l z_i &= G_{l+1}(\alpha)(\lambda^B, z_1, \dots, z_l). \end{aligned}$$

2) Let now \mathcal{L} be arbitrary. Since X is quasi-projective, $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$, for some very ample line bundles \mathcal{L}_i . Using 1) and the analogue of (a_{iii}) for the formal difference, we get:

$$\begin{aligned} G_l(\alpha \cdot \lambda^A)(z_1, \dots, z_l) &= \sum_{i,j} G_l(\alpha \cdot (\lambda_1^A)^i (\lambda_2^A)^j \cdot b_{i,j}^A)(z_1, \dots, z_l) = \\ \sum_{i,j} G_l(\alpha \cdot b_{i,j}^A)((\lambda_1^A)^{\times i}, (\lambda_2^A)^{\times j}, z_1, \dots, z_l) &= G_{l+1}(\alpha)(\lambda_1^B -_B \lambda_2^B, z_1, \dots, z_l) = G_{l+1}(\alpha)(\lambda^B, z_1, \dots, z_l). \end{aligned}$$

□

If $\mathbf{G}(X)$ is defined, we can define $\mathbf{G}(X \times \mathbb{P}^\infty)$ as follows. We have: $A^*(X \times \mathbb{P}^\infty) = A^*(X)[[t]]$, where $t = c_1^A(\mathcal{O}(1))$. For $\alpha(t) = \sum_{i=0}^\infty \alpha_i \cdot t^i$, set:

$$G_l(\alpha(t))(z_1, \dots, z_l) = \sum_i G_{l+i}(\alpha_i)(t^{\times i}, z_1, \dots, z_l) \in B^*[[t]][[z_1, \dots, z_l]],$$

which converges by (a_{ii}). It follows immediately from the definition that (a_{i,ii,iii}) are satisfied.

Lemma 5.6 *Suppose that $\mathbf{G}(X)$ is defined, and satisfies (4). Then the above definition of $\mathbf{G}(X \times \mathbb{P}^\infty)$ satisfies (4) as well.*

Proof: Arbitrary linear bundle \mathcal{L} on $X \times \mathbb{P}^\infty$ has the form $\mathcal{M}(r)$, for some $r \in \mathbb{Z}$, and some linear bundle \mathcal{M} on X . Let $\mu^A = c_1^A(\mathcal{M})$, $\mu^A +_A [r] \cdot_A t = \sum_{i,j} c_{i,j}^A(\mu^A)^i t^j$, and $\gamma \in A^*(X)$. Then, by the definition of $\mathbf{G}(X \times \mathbb{P}^\infty)$, the condition (4) for X , and (a_{iii}) , we get:

$$G_l(\gamma \cdot t^p \cdot (\mu^A +_A [r] \cdot_A t))(z_1, \dots, z_l) = \sum_{i,j} G_{l+p+i+j}(\gamma \cdot c_{i,j}^A)(t^{\times p+j}, (\mu^B)^{\times i}, z_1, \dots, z_l) =$$

$$G_{l+p+1}(\gamma)((\mu^B +_B [r] \cdot_B t), t^{\times p}, z_1, \dots, z_l) = G_{l+1}(\gamma \cdot t^p)((\mu^B +_B [r] \cdot_B t), z_1, \dots, z_l).$$

This extends to arbitrary element $\alpha(t)$ of $A^*(X \times \mathbb{P}^\infty)$ by linearity. \square

Suppose $\mathbf{G}(d-1)$ is defined, and $X \xrightarrow{p_X} \text{Spec}(k)$ is a smooth quasi-projective variety of dimension $\leq d$. Let $D \xrightarrow{d} X$ be divisor with strict normal crossing (see Subsection 7.2) with components $D_i \xrightarrow{\hat{d}_i} D$, and $\gamma = \sum_i (\hat{d}_i)_*(\gamma_i) \in A^{n-l-1}(D)$. Let $\lambda_i^B = c_1^B(\mathcal{O}(D_i))$. Then, let us define:

$$F_l(\gamma|D)(z_1, \dots, z_l) := \sum_i (d_i)_* F_{l+1}(\gamma_i)(\lambda_i^B, z_1, \dots, z_l). \quad (*)$$

Notice, that $\dim(D_i) \leq d-1$, so $\mathbf{G}(D_i)$ is defined. From the equality:

$$F_{l+1}((d_{\{i,j\}/i})_* \delta)(\lambda_i^B, z_1, \dots, z_l) = (d_{\{i,j\}/i})_* F_{l+2}(\delta)(\lambda_j^B, \lambda_i^B, z_1, \dots, z_l)$$

it is clear that our definition does not depend on the presentation of γ as a sum of $(\hat{d}_i)_*(\gamma_i)$. Also it follows from (b_{ii}) that, in the case $\dim(X) \leq (d-1)$ we have:

$$F_l(\gamma|D)(z_1, \dots, z_l) = F_l(d_*(\gamma))(z_1, \dots, z_l).$$

Proposition 5.7 *Suppose, we have a cartesian diagram (7) with X and Y of dimension $\leq d$, and D and E - divisors with strict normal crossing. Then:*

$$f^* F_l(\gamma|D)(z_1, \dots, z_l) = F_l(\bar{f}^*(\gamma)|E)(z_1, \dots, z_l).$$

Proof: From the definition (*) above and the definition of \bar{f}^* (Definition 7.16), it is clear that it is sufficient to treat the case of a smooth D . Let $E = \sum_{j=1}^s m_j \cdot E_j$, $\lambda^A = c_1^A(\mathcal{O}_X(D))$, $\mu_j^A = c_1^A(\mathcal{O}_Y(E_j))$ (and similar for λ^B, μ_j^B). Let us denote $(F_J^{m_1, \dots, m_s})^A \in A[[\mu_1^A, \dots, \mu_s^A]]$ as C_J^A (and similar for B).

Lemma 5.8 *Suppose, we have a cartesian square (7) with $\dim(Y) \leq d$, and D - smooth divisor, E - divisor with strict normal crossing. Then, for any choice of coefficients $C_J^{A,B}$,*

$$F_l(\bar{f}^*(\gamma)|E)(z_1, \dots, z_l) = \sum_{J \subset \{1, \dots, s\}} (e_J)_* \left(C_J^B \cdot F_{l+1}(f_J^*(\gamma))(f_J^*(\lambda^B), z_1, \dots, z_l) \right).$$

Proof: We will denote the 1-st Chern class of the bundle $\mathcal{O}(1)$ on \mathbb{P}^∞ (in both A^* and B^* -theory) by t . Let $\tilde{\mu}_j^A = t +_A \mu_j^A$, and similar for B . Let us denote: $\mu_I^A = \sum_{j \in I} [m_j] \cdot_A \mu_j^A$, and $\tilde{\mu}_I^A = \sum_{j \in I} [m_j] \cdot_A \tilde{\mu}_j^A$ (and similar for μ_I^B). For each subset $\emptyset \neq J \subset \{1, \dots, s\}$, let us denote $\times_{j \in J} \mu_j^B$ as $(\mu^B)^{\times J}$, and $\times_{j \in J} \tilde{\mu}_j^B$ as $(\tilde{\mu}^B)^{\times J}$. From the definition and (b_i) it is clear that both parts do not depend on the choice of coefficients $C_J^{A,B}$. Let us use the standard choice for $C_J^{A,B}$. Recall, that $C_J^A = \frac{\sum_{I \subset J} (-1)^{|J|-|I|} \mu_I^A}{(\mu^A)^J}$. Denote as \tilde{C}_J^A the

analogous coefficients for $\tilde{\mu}_j$. We have:

$$\begin{aligned}
F_l(\bar{f}^*(\gamma)|E)(z_1, \dots, z_l) &= \sum_{J \subset \{1, \dots, s\}} (e_J)_* F_{l+|J|}(f_J^*(\gamma) \cdot C_J^A)((\mu^B)^{\times J}, z_1, \dots, z_l) = \\
&= \sum_{J \subset \{1, \dots, s\}} (e_J)_* \operatorname{Res}_{t=0} \frac{G_{l+|J|}(f_J^*(\gamma) \cdot \tilde{C}_J^A)((\tilde{\mu}^B)^{\times J}, z_1, \dots, z_l) \omega_t^B}{t \cdot (\tilde{\mu}^B)^J \cdot \prod_{i=1}^l z_i} = \\
&= \sum_{J \subset \{1, \dots, s\}} (e_J)_* \operatorname{Res}_{t=0} \frac{G_l(f_J^*(\gamma) \cdot (\sum_{I \subset J} (-1)^{|J|-|I|} \tilde{\mu}_I^A))(z_1, \dots, z_l) \omega_t^B}{t \cdot (\tilde{\mu}^B)^J \cdot \prod_{i=1}^l z_i} = \\
&= \sum_{J \subset \{1, \dots, s\}} (e_J)_* \operatorname{Res}_{t=0} \left(\sum_{I \subset J} (-1)^{|J|-|I|} \frac{G_{l+1}(f_J^*(\gamma))(\tilde{\mu}_I^B, z_1, \dots, z_l)}{t \cdot (\tilde{\mu}^B)^J \cdot \prod_{i=1}^l z_i} \right) \omega_t^B = \\
&= \sum_{J \subset \{1, \dots, s\}} (e_J)_* \operatorname{Res}_{t=0} \left(\sum_{I \subset J} (-1)^{|J|-|I|} \frac{\tilde{\mu}_I^B}{(\tilde{\mu}^B)^J} \cdot \frac{F_{l+1}(f_J^*(\gamma))(\tilde{\mu}_I^B, z_1, \dots, z_l)}{t} \right) \omega_t^B.
\end{aligned}$$

Since, for $L \subset J$, $\sum_{L \subset I \subset J} F_{l+1}(f_L^*(\gamma))(\tilde{\mu}_I^B, z_1, \dots, z_l) \cdot (-1)^{|J|-|I|}$ is divisible by $(\tilde{\mu}^B)^{J/L}$ (this is true for any power series $F(x)$), we have:

$$\begin{aligned}
&(e_{J/L})_* \operatorname{Res}_{t=0} \tilde{C}_L^B \cdot \sum_{L \subset J} \frac{1}{t \cdot (\tilde{\mu}^B)^{J/L}} \cdot \sum_{L \subset I \subset J} (-1)^{|J|-|I|} F_{l+1}(f_J^*(\gamma))(\tilde{\mu}_I^B, z_1, \dots, z_l) \cdot \omega_t^B = \\
&\operatorname{Res}_{t=0} \tilde{C}_L^B \cdot \sum_{L \subset J} \frac{(\mu^B)^{J/L}}{t \cdot (\tilde{\mu}^B)^{J/L}} \cdot \sum_{L \subset I \subset J} (-1)^{|J|-|I|} F_{l+1}(f_L^*(\gamma))(\tilde{\mu}_I^B, z_1, \dots, z_l) \cdot \omega_t^B = \\
&\operatorname{Res}_{t=0} \tilde{C}_L^B \cdot \sum_{L \subset J} \frac{1}{t} \cdot \sum_{L \subset I \subset J} (-1)^{|J|-|I|} F_{l+1}(f_L^*(\gamma))(\tilde{\mu}_I^B, z_1, \dots, z_l) \cdot \omega_t^B,
\end{aligned}$$

we can rewrite our expression as: $\sum_{J \subset \{1, \dots, s\}} (e_J)_* R_J$, where $R_J =$

$$\begin{aligned}
&\operatorname{Res}_{t=0} \left(\sum_{I \subset J} (-1)^{|J|-|I|} \frac{\tilde{\mu}_I^B}{(\tilde{\mu}^B)^J} \cdot \frac{F_{l+1}(f_J^*(\gamma))(\tilde{\mu}_I^B, z_1, \dots, z_l)}{t} + \right. \\
&\quad \sum_{L \subset J} \frac{\tilde{C}_L^B}{t \cdot (\tilde{\mu}^B)^{J/L}} \sum_{L \subset I \subset J} (-1)^{|J|-|I|+1} F_{l+1}(f_J^*(\gamma))(\tilde{\mu}_I^B, z_1, \dots, z_l) + \\
&\quad \left. \sum_{J \subset K} \frac{\tilde{C}_J^B}{t} \sum_{J \subset N \subset K} (-1)^{|K|-|N|} F_{l+1}(f_J^*(\gamma))(\tilde{\mu}_N^B, z_1, \dots, z_l) \right) \omega_t^B = \\
&\operatorname{Res}_{t=0} \left(\sum_{J \subset K} \frac{\tilde{C}_J^B}{t} \sum_{J \subset N \subset K} (-1)^{|K|-|N|} F_{l+1}(f_J^*(\gamma))(\tilde{\mu}_N^B, z_1, \dots, z_l) \right) \omega_t^B = \\
&\operatorname{Res}_{t=0} \left(\frac{\tilde{C}_J^B}{t} F_{l+1}(f_J^*(\gamma))(\tilde{\mu}_{\{1, \dots, s\}}^B, z_1, \dots, z_l) \right) \omega_t^B = C_J^B \cdot F_{l+1}(f_J^*(\gamma))(f_J^*(\lambda^B), z_1, \dots, z_l).
\end{aligned}$$

Thus,

$$F_l(\bar{f}^*(\gamma)|E)(z_1, \dots, z_l) = \sum_{J \subset \{1, \dots, s\}} (e_J)_* \left(C_J^B \cdot F_{l+1}(f_J^*(\gamma))(f_J^*(\lambda^B), z_1, \dots, z_l) \right).$$

□

It remains to observe that our expression is equal to $f^*F_l(\gamma|D)(z_1, \dots, z_l)$, by Proposition 7.21 and (b_i) . \square

Suppose $\mathbf{G}(d-1)$ is defined, and $X \xrightarrow{p_X} \text{Spec}(k)$ is a smooth quasi-projective variety of dimension $\leq d$. Let us define $\mathbf{G}(X)$ as follows. Since A^* is a theory of rational type, by Theorem 4.28, $A^*(X) = A \oplus H(\mathfrak{c})$, where \mathfrak{c} is a bi-complex* of Subsection 4.3.

For the constant part, set: $G_l(p_X^*(\alpha))(z_1, \dots, z_l) := p_X^*G_l(\alpha)(z_1, \dots, z_l)$.

For the $\overline{A}^*(X)$ -part, consider an element of $c_{0,0}$. It is represented by the triple $(\tilde{X} \xrightarrow{\rho} X, V \xrightarrow{v} \tilde{X}, \gamma)$, where ρ is birational projective morphism, V - divisor with strict normal crossing on \tilde{X} , and $\gamma \in A^{n-l-1}(V) \cap \text{im}(\rho^!)$. Recall, that the respective element $\alpha \in \overline{A}^{n-l}(X)$ is $\frac{\rho_*v_*(\gamma)}{\rho_*(1^B)}$. Define:

$$F_l(\alpha)(z_1, \dots, z_l) := \frac{\rho_*F_l(\gamma|V)(z_1, \dots, z_l)}{\rho_*(1^B)}. \quad (**)$$

Notice, that $\dim(V_i) \leq d-1$, so $\mathbf{G}(V_i)$ is defined. So, F_l is well-defined on $c_{0,0}$. Also, for $\dim(X) \leq (d-1)$, it follows from (b_i) and (b_{ii}) that the "new" definition of F_l agrees with the "old" one.

Now we need to check that it is trivial on the images of $d_{1,0}^{\mathfrak{c}}$ and $d_{0,1}^{\mathfrak{c}}$.

Proposition 5.9 *In the above situation, $F_l(\gamma|V) \in \text{image}(\rho^*)$.*

Proof: Consider first the case where ρ is the permitted blow up with smooth centers R_j . Let $R_j \xleftarrow{\varepsilon_j} E_j \xrightarrow{e_j} \tilde{X}$ be the components of the special divisor of ρ . Then, by our condition on V , E_j is among the components V_i of V . Then, by Proposition 7.6, to prove that $F_l(\gamma|V) \in \text{image}(\rho^*)$ we need to show that $e_j^*(F_l(\gamma|V)) \in \text{image}(\varepsilon_j^*)$, for each j . Since V is a divisor with strict normal crossing on \tilde{X} , and E_j is a component of it, for any other component V_i of V , the left cartesian diagram below is transversal:

$$\begin{array}{ccc} H_{i,j} & \xrightarrow{u_{i,j}} & E_j \\ h_{i,j} \downarrow & & \downarrow e_j \\ V_i & \xrightarrow{v_i} & \tilde{X}. \end{array} \quad \begin{array}{ccccc} V & \xrightarrow{v} & \tilde{X} & \xleftarrow{e_j} & E_j \\ \rho_V \downarrow & & \downarrow \rho & & \downarrow \varepsilon_j \\ Z & \xrightarrow{z} & X & \xleftarrow{r_j} & R_j \end{array}$$

By (the very simple case of) Proposition 5.7, $e_j^*F_l(\gamma|V_i) = F_l(h_{i,j}^*(\gamma_i)|H_{i,j})$. But $\dim(H_{i,j}), \dim(V_i), \dim(E_j) \leq (d-1)$. Thus,

$$F_l(h_{i,j}^*(\gamma_i)|H_{i,j}) = F_l((u_{i,j})_*h_{i,j}^*(\gamma_i)) = F_l(e_j^*(v_i)_*(\gamma_i)).$$

And the same is true for the E_j -component: $e_j^*F_l(\gamma|E_j) = F_l(e_j^*(e_j)_*(\gamma_j))$. Hence, by [8, Theorem 6.6.6 (2)(a)],

$$\begin{aligned} e_j^*F_l(\gamma|V) &= \sum_i e_j^*F_l(\gamma_i|V_i) = \sum_i F_l(e_j^*(v_i)_*(\gamma_i)) = F_l(e_j^*v_*(\gamma)) = F_l(e_j^*v_*\rho^!(\beta)) = F_l(e_j^*\rho^*z_*(\beta)) = \\ &F_l(\varepsilon_j^*r_j^*z_*(\beta)) = \varepsilon_j^*F_l(r_j^*z_*(\beta)). \end{aligned}$$

So, $F_l(\gamma|V) \in \text{image}(\rho^*)$.

Let now ρ be just projective bi-rational map. Let $Z \subset X$ be closed subset $\rho(V)$ outside which ρ is an isomorphism. Then there exists a permitted blow up $\rho' = \rho \circ \pi$ with centers over Z . Let $V' = \pi^*(V)$. By Proposition 5.7, $\pi^*F_l(\gamma|V) = F_l(\pi_V^*(\gamma)|V')$, which is in the $\text{image}(\pi^*\rho^*)$ by the case proven. Since π^* is injective, $F_l(\gamma|V) \in \text{image}(\rho^*)$. \square

The 1-st part of $(d_{1,0}^c)$: Let

$$\begin{array}{ccc} V' & \xrightarrow{v'} & \widetilde{X} \\ \pi_V \downarrow & & \downarrow \pi \\ V & \xrightarrow{v} & \widetilde{X} \end{array}$$

be the cartesian square, with V and V' divisors with strict normal crossing, with π the blow up permitted w.r.to V . We need to check that the triples: $(\widetilde{X} \xrightarrow{\rho} X, V \xrightarrow{v} \widetilde{X}, \gamma)$ and $(\widetilde{X} \xrightarrow{\rho\pi} X, V' \xrightarrow{v'} \widetilde{X}, \pi_V^*(\gamma))$ produce the same result. From Propositions 5.7 and 5.9,

$$\frac{\rho_*\pi_*F_l(\pi_V^*\gamma|V')(z_1, \dots, z_l)}{\rho_*\pi_*(1^B)} = \frac{\rho_*\pi_*\pi^*F_l(\gamma|V)(z_1, \dots, z_l)}{\rho_*\pi_*\pi^*(1^B)} = \frac{\rho_*F_l(\gamma|V)(z_1, \dots, z_l)}{\rho_*(1^B)}$$

Thus, F_l is trivial on the image of the 1-st part of $(d_{1,0}^c)$.

The 2-nd part of $(d_{1,0}^c)$: It follows immediately from the definition of $F_l(\gamma|V)$ that F_l is trivial on the image of the 2-nd part of $(d_{1,0}^c)$.

$(d_{0,1}^c)$: Let $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho} X \times \mathbb{P}^1$ be the blow up permitted w.r.to $X \times \{0\}$ and $X \times \{1\}$ having no centers over 0, or 1, which is an isomorphism outside the strict normal crossing divisor W , and for each component S of W , $S_0 = s^{-1}(X \times \{0\}) \xrightarrow{i_0} S$ and $S_1 = s^{-1}(X \times \{1\}) \xrightarrow{i_1} S$ are divisors with strict normal crossing. Let $X_0 = \rho^{-1}(X \times \{0\})$, and $X_1 = \rho^{-1}(X \times \{1\})$. Let $\delta = \sum_S \delta_S \in A^{n-l-1}(W) \cap im(\rho^!)$. We need to show that F_l takes the same values on the triples

$$(X_0 \xrightarrow{\rho_0} X, W_0 \xrightarrow{w_0} X_0, \sum_S i_0^*(\delta_S)) \quad \text{and} \quad (X_1 \xrightarrow{\rho_1} X, W_1 \xrightarrow{w_1} X_1, \sum_S i_1^*(\delta_S)).$$

Let S be some component of W , and $S_{0,k} \xrightarrow{i_{0,k}} S_0$, $S_{1,k} \xrightarrow{i_{1,k}} S_1$ be the components of S_0 , S_1 . Let $\widetilde{S} \xrightarrow{p} S$ be some projective bi-rational morphism, which is an isomorphism outside some divisor with normal crossing $H \xrightarrow{h} \widetilde{S}$, and such that $p \circ h(H)$ does not contain components of S_0 and S_1 . Let $\lambda^{A,B} = c_1^{A,B}(\mathcal{O}_{\widetilde{X \times \mathbb{P}^1}}(S))$, and $\mu_{0,k}^{A,B} = c_1^{A,B}(\mathcal{O}_{X_0}(S_{0,k}))$. Let $\gamma = p^!(u) \in A^{n-l-2}(H)$, and $\beta \in A^{n-l-1}(S)$ be the image of u (in particular, $h_*(\gamma) = p^*(\beta)$). Let us denote:

$$\widetilde{F}_l(\beta|S)(z_1, \dots, z_l) := s_* \left(\frac{p_* F_{l+1}(\gamma|H)(p^*(\lambda^B), z_1, \dots, z_l)}{p_*(1)} \right)$$

Lemma 5.10 *In the above situation,*

$$\widetilde{i}_0^*(\widetilde{F}_l(\beta|S)) = F_l(i_0^*(\beta)|S_0) \quad \text{and} \quad \widetilde{i}_1^*(\widetilde{F}_l(\beta|S)) = F_l(i_1^*(\beta)|S_1)$$

Proof: It is sufficient to treat the S_0 case. By our condition, $p \circ h(H)$ itersects each component of S_0 and S_1 in positive codimension.

Consider one of these components $S_{0,k}$. By Theorems 8.3 and 8.4, we can find a permitted blow up $\widetilde{S}_{0,k} \xrightarrow{p_{0,k}} S_{0,k}$, which fits into the diagram:

$$\begin{array}{ccccccc} H_{0,k} & \xrightarrow{h_{0,k}} & \widetilde{S}_{0,k} & \xrightarrow{p_{0,k}} & S_{0,k} & \xrightarrow{s_{0,k}} & X_0 \\ i_{0,H} \downarrow & & \widetilde{i}_{0,k} \downarrow & & \downarrow i_{0,k} & & \downarrow \widetilde{i}_0 \\ H & \xrightarrow{h} & \widetilde{S} & \xrightarrow{p} & S & \xrightarrow{s} & \widetilde{X \times \mathbb{P}^1} \end{array}$$

where the left square is cartesian, and $H_{0,k}$ is a divisor with strict normal crossing on $\widetilde{S}_{0,k}$.

Lemma 5.11 *Let*

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{\tilde{j}} & \tilde{S} \\ q \downarrow & & \downarrow p \\ T & \xrightarrow{j} & S \end{array}$$

be commutative diagram with p and q - projective bi-rational. Let $x \in \text{im}(p^*)$. Then:

$$\frac{q_*(\tilde{j}^*(x))}{q_*(1)} = j^*\left(\frac{p_*(x)}{p_*(1)}\right).$$

Proof: Let $x = p^*(y)$. Then

$$q_*(\tilde{j}^*p^*(y)) \cdot j^*p_*(1) = q_*q^*j^*(y) \cdot j^*p_*(1) = q_*(1) \cdot j^*(y) \cdot j^*p_*(1) = q_*(1) \cdot j^*(p_*p^*(y)),$$

which implies what we need. \square

By Proposition 5.9, $F_{l+1}(\gamma|H)(p^*(\lambda^B), z_1, \dots, z_l) \in \text{im}(p^*)$. Then, by Lemma 5.11, Proposition 5.7 and (b_i) , we have:

$$\begin{aligned} i_{0,k}^*\left(\frac{p_*F_{l+1}(\gamma|H)(p^*(\lambda^B), z_1, \dots, z_l)}{p_*(1)}\right) &= \frac{(p_{0,k})_*(\tilde{i}_{0,k}^*F_{l+1}(\gamma|H)(p^*(\lambda^B), z_1, \dots, z_l))}{(p_{0,k})_*(1)} = \\ &= \frac{(p_{0,k})_*(F_{l+1}(\tilde{i}_{0,k}^*h_*(\gamma))(\tilde{i}_{0,k}^*p^*(\lambda^B), z_1, \dots, z_l))}{(p_{0,k})_*(1)} = \frac{(p_{0,k})_*(F_{l+1}(p_{0,k}^*i_{0,k}^*(\beta))(p_{0,k}^*i_{0,k}^*(\lambda^B), z_1, \dots, z_l))}{(p_{0,k})_*(1)} = \\ &= F_{l+1}(i_{0,k}^*(\beta))(i_{0,k}^*(\lambda^B), z_1, \dots, z_l). \end{aligned}$$

We can assume that coefficients $C_J^{A,B}$ are chosen to be zero, for $|J| > 1$. Then, by Proposition 7.21, and Lemma 5.8

$$\begin{aligned} \tilde{i}_0^*(\tilde{F}_l(\beta|S)(z_1, \dots, z_l)) &= \tilde{i}_0^*s_*\left(\frac{p_*F_{l+1}(\gamma|H)(p^*(\lambda^B), z_1, \dots, z_l)}{p_*(1)}\right) = \\ &= \sum_k (s_{0,k})_*(C_k^B \cdot F_{l+1}(i_{0,k}^*(\beta))(i_{0,k}^*(\lambda^B), z_1, \dots, z_l)) = \sum_k (s_{0,k})_*F_{l+1}(i_{0,k}^*(\beta) \cdot C_k^A)(\mu_{0,k}^B, z_1, \dots, z_l) = \\ &= F_l(i_0^*(\beta)|S_0)(z_1, \dots, z_l). \end{aligned}$$

\square

Let $S \xrightarrow{\pi_S^S} \text{Spec}(k)$ be the natural projection. Denote:

$$\tilde{F}_l(1|S)(z_1, \dots, z_l) := s_*(\pi_S^*F_{l+1}(1)(s^*(\lambda^B), z_1, \dots, z_l)).$$

Lemma 5.12 *In the above situation,*

$$\tilde{i}_0^*(\tilde{F}_l(1|S)) = F_l(i_0^*(1)|S_0) \quad \text{and} \quad \tilde{i}_1^*(\tilde{F}_l(1|S)) = F_l(i_1^*(1)|S_1)$$

Proof: We treat S_0 only. By Proposition 7.21 and Lemma 5.8, we have:

$$\begin{aligned} \tilde{i}_0^*s_*(\pi_S^*F_{l+1}(1)(s^*(\lambda^B), z_1, \dots, z_l)) &= \sum_k (s_{0,k})_*(C_k^B \cdot i_{0,k}^*\pi_S^*F_{l+1}(1)(i_{0,k}^*s^*(\lambda^B), z_1, \dots, z_l)) = \\ &= \sum_k (s_{0,k})_*(C_k^B \cdot F_{l+1}(1)(s_{0,k}^*\tilde{i}_0^*(\lambda^B), z_1, \dots, z_l)) = \sum_k (s_{0,k})_*F_{l+1}(C_k^A)(\mu_{0,k}^B, z_1, \dots, z_l) = \\ &= F_l(i_0^*(1)|S_0)(z_1, \dots, z_l). \end{aligned}$$

\square

Proposition 5.13 *In the above situation,*

$$\frac{(\rho_0)_* F_l(\sum_S i_0^*(\delta_S)|W_0)(z_1, \dots, z_l)}{(\rho_0)_*(1)} = \frac{(\rho_1)_* F_l(\sum_S i_1^*(\delta_S)|W_1)(z_1, \dots, z_l)}{(\rho_1)_*(1)}.$$

Proof:

Lemma 5.14 *Let S be quasi-projective variety, and T be some divisor on it. Then any element $\beta \in \overline{A}_*(S)$ can be represented by an element from $A_*(Z)$, where $Z \rightarrow S$ is a closed subscheme containing no components of T .*

Proof: Since A^* is obtained from Algebraic Cobordism theory by the change of coefficients, it is sufficient to treat the case of $A^* = \Omega^*$, and $\beta = \pi_*(1_{\tilde{T}_i})$, where $\tilde{T}_i \xrightarrow{\pi} T_i$ is the resolution of singularities of some component of T . Since S is quasi-projective, $1_{T_i}^{CH}$ can be represented as $\sum_k 1_{R_k}^{CH}$, where R_k are irreducible divisors different from components of T . Taking resolutions $\tilde{R}_k \xrightarrow{\rho_k} R_k$, we have: $\pi_*(1_{\tilde{T}_i}) = \sum_k (\rho_k)_*(1_{\tilde{R}_k}) + \gamma$, where γ has support of codimension ≥ 2 . \square

Lemma 5.14 together with Theorem 8.4 implies that any element $\beta \in \overline{A}^{n-l-1}(S) = H(\mathfrak{b})$ can be represented by an element $(\tilde{S} \xrightarrow{p} S, H \xrightarrow{h} \tilde{S}, x)$, where p is projective bi-rational, isomorphism outside $Z \subset S$, where Z is a closed subscheme containing no components of S_0 and S_1 , $H = p^{-1}(Z)$ is a divisor with strict normal crossing on \tilde{S} , and $x \in A^{n-l-2}(H)$. Then the respective element in $H(\mathfrak{c})$ will be $(\tilde{S} \xrightarrow{p} S, H \xrightarrow{h} \tilde{S}, p^!(p_H)_*(x))$. Taking $\gamma = p^!(p_H)_*(x)$ and $\beta = p_* h_*(x)$ as above, we obtain from Lemma 5.10 that

$$\tilde{i}_0^*(\tilde{F}_l(\beta|S)) = F_l(i_0^*(\beta)|S_0) \quad \text{and} \quad \tilde{i}_1^*(\tilde{F}_l(\beta|S)) = F_l(i_1^*(\beta)|S_1).$$

And the same is true for $\beta = 1$, by Lemma 5.12. Since ρ has no centers over 0 and 1, the following cartesian diagram is transversal:

$$\begin{array}{ccccc} X_0 & \xrightarrow{\tilde{i}_0} & \widetilde{X \times \mathbb{P}^1} & \xleftarrow{\tilde{i}_1} & X_1 \\ \rho_0 \downarrow & & \rho \downarrow & & \downarrow \rho_1 \\ X \times \{0\} & \xrightarrow{i_0} & X \times \mathbb{P}^1 & \xleftarrow{i_1} & X \times \{1\}. \end{array}$$

This implies that, for any $\delta \in A^{n-l-1}(W)$,

$$\begin{aligned} \frac{(\rho_0)_* F_l(\sum_S i_0^*(\delta_S)|W_0)(z_1, \dots, z_l)}{(\rho_0)_*(1)} &= i_0^* \left(\frac{\rho_* \tilde{F}_l(\delta|W)(z_1, \dots, z_l)}{\rho_*(1)} \right) = \\ i_1^* \left(\frac{\rho_* \tilde{F}_l(\delta|W)(z_1, \dots, z_l)}{\rho_*(1)} \right) &= \frac{(\rho_1)_* F_l(\sum_S i_1^*(\delta_S)|W_1)(z_1, \dots, z_l)}{(\rho_1)_*(1)}. \end{aligned}$$

\square

It follows from Proposition 5.13 that F_l is trivial on the image of $d_{0,1}^c$. Thus, we obtain:

Proposition 5.15 *Suppose, $\mathbf{G}(d-1)$ is defined, and X is smooth quasi-projective variety of dimension d . Then the above definition of F_l is well-defined and satisfies the conditions of $\mathbf{G}(X)$.*

Proof: We already have proven that F_l is well-defined on $A^{n-l}(X)$. The conditions (a_i) , (a_{ii}) follow immediately from the definition of F_l . As for (a_{iii}) , it is clearly sufficient to treat the case, where $V \xrightarrow{v} X$ is a smooth divisor, and $\alpha = v_*(\gamma)$. Let $\lambda^{A,B} = c_1^{A,B}(\mathcal{O}_X(V))$. Then

$$G_l(v_*(\gamma))(x +_B y, z_2, \dots, z_l) = v_* \operatorname{Res}_{t=0} \frac{G_{l+1}(\gamma)(t +_B \lambda^B, x +_B y, z_2, \dots, z_l) \omega_t^B}{(t +_B \lambda^B) \cdot t} =$$

$$v_* \operatorname{Res}_{t=0} \sum_{i,j} \frac{G_{l+i+j}(\gamma \cdot a_{i,j}^A)(t +_B \lambda^B, x^{\times i}, y^{\times j}, z_2, \dots, z_l) \omega_t^B}{(t +_B \lambda^B) \cdot t} = \sum_{i,j} G_{l+i+j-1}(v_*(\gamma) \cdot a_{i,j}^A)(x^{\times i}, y^{\times j}, z_2, \dots, z_l).$$

□

Proposition 5.16 *Suppose, $\mathbf{G}(d-1)$ is defined. Then it extends to $\mathbf{G}(d)$.*

Proof: Above we have defined $\mathbf{G}(X)$, for each X of dimension $\leq d$, which, in the case of $\dim(X) \leq (d-1)$, coincides with the "old" definition. It remains to check the conditions $(b_{i,ii})$. Let X, Y be smooth varieties of dimension $\leq d$. For constant $\alpha \in A^{n-l}(X)$, (b_i) is evident. Consider now the case $\alpha \in \overline{A}^{n-l}(X)$. Let $\tilde{X} \xrightarrow{\rho_X} X$ be the projective-birational map, which is an isomorphism outside the strict normal crossing divisor $V_X \xrightarrow{v_X} \tilde{X}$, and $\gamma \in A^{n-l-1}(V_X) \cap \rho_X^!$. We can assume that $\alpha = \frac{(\rho_X)_*(v_X)_*(\gamma)}{(\rho_X)_*(1)}$. By Theorems 8.3 and 8.4, we can find the projective bi-rational map $\tilde{Y} \xrightarrow{\rho_Y} Y$, which fits into the commutative diagram with the left square cartesian, and $V_Y \xrightarrow{v_Y} \tilde{Y}$ - the divisor with strict normal crossing:

$$\begin{array}{ccccc} V_Y & \xrightarrow{v_Y} & \tilde{Y} & \xrightarrow{\rho_Y} & Y \\ f_V \downarrow & & \downarrow f & & \downarrow f \\ V_X & \xrightarrow{v_X} & \tilde{X} & \xrightarrow{\rho_X} & X. \end{array}$$

By Proposition 5.9, Lemma 5.11, Proposition 5.7, and Proposition 7.21,

$$f^* F_l \left(\frac{(\rho_X)_*(v_X)_*(\gamma)}{(\rho_X)_*(1)} \right) = f^* \left(\frac{(\rho_X)_* F_l(\gamma|V_X)}{(\rho_X)_*(1)} \right) = \frac{(\rho_Y)_* \tilde{f}^* F_l(\gamma|V_X)}{(\rho_Y)_*(1)} =$$

$$\frac{(\rho_Y)_* F_l(f_V^*(\gamma)|V_Y)}{(\rho_Y)_*(1)} = F_l \left(\frac{(\rho_Y)_*(v_Y)_* f_V^*(\gamma)}{(\rho_Y)_*(1)} \right) = F_l \left(\frac{(\rho_Y)_* \tilde{f}^*(v_X)_*(\gamma)}{(\rho_Y)_*(1)} \right) =$$

$$F_l \left(f^* \left(\frac{(\rho_X)_*(v_X)_*(\gamma)}{(\rho_Y)_*(1)} \right) \right). \quad \text{This proves } (b_i).$$

Let now $X \xrightarrow{j} Y$ be regular embedding of codimension r with normal bundle \mathcal{N}_j , with $\dim(Y) \leq d$. Consider the blow-up diagram:

$$\begin{array}{ccc} E & \xrightarrow{\tilde{j}} & \tilde{Y} \\ \varepsilon \downarrow & & \downarrow \pi \\ X & \xrightarrow{j} & Y, \end{array}$$

where $E = \mathbb{P}_X(\mathcal{N}_j)$, and $\mathcal{N}_{\tilde{j}} = \mathcal{O}(-1)$. Let $\mathcal{M} = \varepsilon^* \mathcal{N}_j / \mathcal{O}(-1)$, $\nu_1^{A,B}, \dots, \nu_{r-1}^{A,B}$ - be roots of \mathcal{M} , $\zeta^{A,B}$ - root of $\mathcal{O}(-1)$, and $\alpha \in A^{n-l-r}(X)$. Then, by the already proven (b_i) , the Excess Intersection Formula

(Proposition 7.1), the definition of $\mathbf{G}(\widetilde{Y})$, Lemma 5.5, again (b_i) , and Proposition 7.1 again, we get:

$$\begin{aligned} \pi^* F_l(j_*(\alpha))(z_1, \dots, z_l) &= F_l(\pi^* j_*(\alpha))(z_1, \dots, z_l) = F_l(\widetilde{j}_*(c_{r-1}^A(\mathcal{M}) \cdot \varepsilon^*(\alpha)))(z_1, \dots, z_l) = \\ \widetilde{j}_* F_{l+1}(c_{r-1}^A(\mathcal{M}) \cdot \varepsilon^*(\alpha))(\zeta^B, z_1, \dots, z_l) &= \widetilde{j}_* \left(\prod_{i=1}^{r-1} \nu_i^B \cdot F_{l+1}(\varepsilon^*(\alpha))(\zeta^B, \nu_1^B, \dots, \nu_{r-1}^B, z_1, \dots, z_l) \right) = \\ \widetilde{j}_* \left(c_{r-1}^B(\mathcal{M}) \cdot \varepsilon^*(F_{l+1}(\alpha)(\mu_1^B, \dots, \mu_r^B, z_1, \dots, z_l)) \right) &= \pi^* j_* F_{l+1}(\alpha)(\mu_1^B, \dots, \mu_r^B, z_1, \dots, z_l). \end{aligned}$$

And since π^* is injective, we obtain (b_{ii}) . \square

Proof of Theorem 5.1: By the conditions of the Theorem, $\mathbf{G}(0)$ is defined. Then it follows from Proposition 5.16 that it can be extended to $\mathbf{G}(\infty)$. Consider now $G_0 : A^n(X) \rightarrow B^m(X)$, for all $X \in \mathbf{Sm}_k$. By (b_i) , this is an additive operation. It remains to see that it extends the original $A^n((\mathbb{P}^\infty)^{\times l}) \xrightarrow{G} B^m((\mathbb{P}^\infty)^{\times l})$. From commutativity with the pull-backs for partial diagonals, it is sufficient to compare the results on $\alpha \cdot \prod_{i=1}^l z_i^A \in A^n((\mathbb{P}^\infty)^{\times l})$, where $\alpha \in A^{n-l}$, and $z_i^A = c_1^A(\mathcal{O}(1)_i)$. Let $j : (\mathbb{P}^\infty)^{\times l} \rightarrow (\mathbb{P}^\infty)^{\times l}$ be the product of hyperplane section embeddings. Then $G_0(j_*(\alpha)) = G_l(\alpha)(z_1^B, \dots, z_l^B) = G(\alpha \cdot \prod_{i=1}^l z_i^A)$, by (b_{ii}) and the definition of $\mathbf{G}(\text{Spec}(k))$. Thus, G_0 extends the original transformation on products of projective spaces. The uniqueness follows from Proposition 3.15. \square

We will also need the following multiplicative version:

Proposition 5.17 *Suppose, in the situation of Theorem 5.1, the original transformation $A^*((\mathbb{P}^\infty)^{\times l}) \xrightarrow{G} B^*((\mathbb{P}^\infty)^{\times l})$ commutes with the external products of projective spaces. Then the resulting operation G is multiplicative.*

Proof: We need to prove that G respects external products of varieties:

$$G_{l+m}(\alpha \times \beta)(x_1, \dots, x_l, y_1, \dots, y_m) = G_l(\alpha)(x_1, \dots, x_l) \times G_m(\beta)(y_1, \dots, y_m),$$

for any $\alpha \in A^*(X)$, $\beta \in A^*(Y)$. We first prove it for the case $Y = \text{Spec}(k)$ by induction on the dimension of X . The base and the case where α is constant follow from our condition. In the case $\alpha \in \overline{A}^*(X)$, we can find a projective bi-rational morphism $\widetilde{X} \xrightarrow{\rho} X$ such that $\rho^*(\alpha)$ is supported on some divisor with strict normal crossing. Since ρ^* is injective, without loss of generality, we can assume that $\alpha = v_*(\alpha')$, where $V \xrightarrow{v} X$ is a smooth divisor. Let $\lambda^B = c_1^B(\mathcal{O}_X(V))$. Then

$$\begin{aligned} G_{l+m}(\alpha \times \beta)(x_1, \dots, x_l, y_1, \dots, y_m) &= (v \times id)_* G_{l+m+1}(\alpha' \times \beta)(\lambda^B, x_1, \dots, x_l, y_1, \dots, y_m) = \\ (v \times id)_* (G_{l+1}(\alpha')(\lambda^B, x_1, \dots, x_l) \times G_m(\beta)(y_1, \dots, y_m)) &= G_l(\alpha)(x_1, \dots, x_l) \times G_m(\beta)(y_1, \dots, y_m), \end{aligned}$$

which proves the induction step. Now, by the induction on the $\dim(Y)$, using similar arguments, we prove the general case. \square

6 Applications

6.1 Unstable operations in Algebraic Cobordism

As a first application of our main result (Theorem 5.1), let us finish the description of unstable operations in Algebraic Cobordism:

Theorem 6.1 *The correspondence: $G_\psi \leftrightarrow \psi$, where $(G_\psi)_\mathbb{Q}$ is the map: $\Omega^* \xrightarrow{S_{L-N}^{Tot}} \Omega^*[\bar{b}] \xrightarrow{\otimes \psi} \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}$ identifies the set of (unstable) additive operations $\Omega^n \rightarrow \Omega^m$ with the subset of $\text{Hom}_{\mathbb{L}}(\mathbb{L}[\bar{b}], \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{Q})_{(m-n)}$, corresponding to exactly those linear combinations S of the Landweber-Novikov operations which satisfy the integrality conditions: $S(\Omega^n((\mathbb{P}^\infty)^{\times r})) \subset \Omega^m((\mathbb{P}^\infty)^{\times r})$, for all r .*

Proof: It follows immediately from Theorems 3.19 and 5.1. \square

If A^* is a theory of rational type, and B^* is any theory in the sense of Definition 2.1, then (unstable) operations $A^n \rightarrow B^m$ can be described as follows.

Theorem 6.2 *Let A^* be theory of rational type, and B^* be any theory. Then there is 1-to-1 correspondence between the set of (unstable) operations $A^n \xrightarrow{G} B^m$ and the set consisting of the following data $\{G_l, l \in \mathbb{N}\}$:*

$$G_l \in \text{Hom}_{\mathbb{Z}\text{-lin}}(A^{n-l}, B[[z_1, \dots, z_l]]_{(m)}) \quad \text{satisfying:}$$

- (a_i) G_l is symmetric with respect to \mathfrak{S}_l ;
- (a_{ii}) $G_l(\alpha) = \prod_{i=1}^l z_i \cdot F_l(\alpha)$, for some $F_l(\alpha) \in B[[z_1, \dots, z_l]]_{(m-l)}$.
- (a_{iii}) $G_l(\alpha)(x +_B y, z_2, \dots, z_l) = \sum_{i,j \geq 1} G_{i+j+l-1}(\alpha \cdot a_{i,j}^A)(x^{\times i}, y^{\times j}, z_2, \dots, z_l)$,
where $a_{i,j}^A$ are the coefficients of the FGL of A^* .

Proof: It follows immediately from Proposition 3.15, Theorem 5.1, and the discussion right after it. \square

With any data $\{G_l, l \in \mathbb{N}\}$ one can associate the data $\{\tilde{G}_l, l \in \mathbb{N}\}$, where $\tilde{G}_l : A^{n-l} \rightarrow B^{m-l}$ is the constant term of the $F_l : A^{n-l} \rightarrow B[[z_1, \dots, z_l]]_{(m-l)}$. We have:

Proposition 6.3 *If B has no torsion, then $\{\tilde{G}_l, l \in \mathbb{N}\}$ carries the same information as $\{G_l, l \in \mathbb{N}\}$.*

Proof: Let $F_l(\alpha) = \sum_I h_{l,\bar{i}}(\alpha) \bar{z}^{\bar{i}}$. Let us prove by induction on the degree of \bar{i} (simultaneously for all l) that $h_{l,\bar{i}}$ is determined by $h_{r,\bar{0}}$, for all r . Base is evident. Consider the equation (a_{iii}). Let $\bar{z}^{\bar{i}} = z_1^{i_1} \dots z_l^{i_l}$, where $i_1 \geq i_2 \geq \dots \geq i_l$ (by (a_i), it is sufficient to treat this case). Using the fact that $x +_B y = x + y + \text{higher terms}$, comparing coefficients at $x^{i_1} y z_2^{i_2+1} \dots z_l^{i_l+1}$, we get that $i_1 \cdot h_{l,\bar{i}}(\alpha)$ is expressible in terms of $h_{r,\bar{j}}(\text{something})$, for $|\bar{j}| < |\bar{i}|$, and all r . Since B has no torsion, $h_{l,\bar{i}}(\alpha)$ is determined by these smaller terms. \square

Corollary 6.4 *Let A^* be a theory satisfying (CONST), and B^* be any theory in the sense of Definition 2.1 with B torsion-free. Then an additive (unstable) operation $A^n \rightarrow B^m$ is determined by its action on the image of $(j_l)_*$, for all l , where $j_l : \text{Spec}(k) \rightarrow (\mathbb{P}^1)^l$ is an embedding of a rational point.*

Proof: This follows from Propositions 3.15 and 6.3. \square

The following example shows that if B has torsion, then $\{\tilde{G}_l, l \in \mathbb{N}\}$ does not determine $\{G_l, l \in \mathbb{N}\}$.

Example 6.5 *Consider $A^* = B^* = \text{CH}^*/p$, p -prime. Then $F_B(x, y) = x + y$ is additive, and $A^{n-l} = 0$, for $l \neq n$. Thus, $G_l = 0$, for $l \neq n$, and the only conditions on G_n are: symmetry and additivity. Thus, $G_n(z_1, \dots, z_n)$ is an arbitrary symmetric polynomial with \mathbb{Z}/p -coefficients of degree m containing monomials where each z_i enters in degree p^{r_i} , where $r_i \geq 0$. And \tilde{G}_n is the coefficient at $z_1 \dots z_n$ (so, it is zero if $n \neq m$, and an element of \mathbb{Z}/p , if $n = m$). Of course, it does not determine G_n .*

In the case of Chow groups modulo p we can describe all the operations explicitly. These appear to be essentially stable, and so expressible in term of Steenrod operations (defined by V.Voevodsky [22] and P.Brosnan [3]).

Theorem 6.6 *Any additive operation $\mathrm{CH}^n/p \rightarrow \mathrm{CH}^m/p$ extends uniquely to a stable operation. The \mathbb{F}_p -vector space of stable operations has a basis consisting of Steenrod operations $S^{\bar{k}}$, where $\bar{k} = (k_1, \dots, k_s)$ is a partition with $k_i = p^{r_i} - 1$, $r_i \geq 0$.*

Proof: In the example 6.5 we saw that any additive operation $\mathrm{CH}^n/p \xrightarrow{G} \mathrm{CH}^m/p$ is determined by some symmetric polynomial $F_n(z_1, \dots, z_n)$ of degree $(m - n)$, where each variable z_i enters in degree $p^{r_i} - 1$, for some $r_i \geq 0$. The value of G on the class $x_n = \prod_{i=1}^n h_i \in \mathrm{CH}^n((\mathbb{P}^\infty)^{\times n})$ is equal to $x_n \cdot F_n(h_1, \dots, h_n)$, which coincides with the value of the (stable!) Steenrod operation S^{F_n} . Since $\mathrm{CH}^{n-l}(\mathrm{Spec}(k))/p = 0$, for $l \neq n$, these two operations $\mathrm{CH}^n/p \rightarrow \mathrm{CH}^m/p$ coincide on \mathbf{Sm}_k , by Proposition 3.15, and the same considerations imply uniqueness of stable extension. Clearly, the \mathbb{F}_p -vector space of mentioned polynomials F_n has a basis consisting of the symmetrizations of monomials corresponding to partitions as above. \square

Remark 6.7 *In particular, Theorem 6.6 provides another construction of Steenrod operations in Chow groups.*

6.2 Multiplicative operations between theories of rational type

The following result reduces the study of multiplicative operations on theories of rational type to the study of morphisms of FGLs (recall, that such theories are in 1-to-1 correspondence with FGLs).

Theorem 6.8 *Let A^* be theory of rational type, and B^* be any theory in the sense of Definition 2.1. The assignment $G \leftrightarrow (\varphi_G, \gamma_G)$ defines a 1-to-1 correspondence between the multiplicative operations $A^* \xrightarrow{G} B^*$ and the homomorphisms $(A, F_A) \rightarrow (B, F_B)$ of the respective formal group laws.*

Proof: Any multiplicative operation G defines the homomorphism $(\varphi_G, \gamma_G) : (A, F_A) \rightarrow (B, F_B)$ of FGLs. On the other hand, any homomorphism (φ, γ) defines the transformation $A^*((\mathbb{P}^\infty)^{\times r}) \xrightarrow{H} B^*((\mathbb{P}^\infty)^{\times r})$ by the rule:

$$H(f(z_1^A, \dots, z_r^A)) := \varphi(f)(\gamma(z_1^B), \dots, \gamma(z_r^B)),$$

where $f \in A[[z_1^A, \dots, z_r^A]] = A^*((\mathbb{P}^\infty)^{\times r})$. Clearly, this transformation commutes with the pull-backs for the action of \mathfrak{S}_l , and for partial diagonals. As for partial Segre embeddings, let $\mathrm{Seg} = (\mathrm{Segre} \times \mathrm{id}^{\times(r-1)})$. Then we have:

$$\begin{aligned} \mathrm{Seg}^* f(z_1^A, \dots, z_r^A) &= f(F_A(x^A, y^A), z_2^A, \dots, z_r^A), \text{ while} \\ \mathrm{Seg}^* \varphi(f)(\gamma(z_1^B), \dots, \gamma(z_r^B)) &= \varphi(f)(\gamma(F_B(x^B, y^B)), \gamma(z_2^B), \dots, \gamma(z_r^B)). \end{aligned}$$

Since $\varphi(F_A)(\gamma(x^B), \gamma(y^B)) = \gamma(F_B(x^B, y^B))$, we get that our transformation commutes with the pull-backs for Segre embeddings as well. Thus, it extends to a unique operation $A^* \xrightarrow{H} B^*$. Since our transformation on $(\mathbb{P}^\infty)^{\times r}$ commutes with the external products of projective spaces, it follows from Proposition 5.17 that the resulting operation will be multiplicative. It follows from Proposition 3.15 that the above two assignments are inverse to each other. \square

Consider now the case where $A^* = \Omega^*$. We can extend the Theorem 3.8.

Theorem 6.9 *Let B^* be any theory in the sense of Definition 2.1, and $b_0 \in B$ be not a zero-divisor. Let $\gamma = b_0x + b_1x^2 + b_2x^3 + \dots \in B[[x]]$. Then there exists a multiplicative operation $\Omega^* \xrightarrow{G} B^*$ with $\gamma_G = \gamma$ if and only if the shifted FGL $F_B^\gamma \in B[b_0^{-1}][[x, y]]$ has coefficients in B (that is, has no denominators). In this case, such an operation is unique.*

Proof: Since $\varphi_G(F_\Omega)(\gamma(x^B), \gamma(y^B)) = \gamma(F_B(x^B, y^B))$, and $\varphi_G(F_\omega)$ has coefficients in B , the above condition is necessary. On the other hand, if $\gamma(F_B(\gamma^{-1}(u), \gamma^{-1}(v)))$ has coefficients in B , by universality of the FGL (\mathbb{L}, F_Ω) , we get a ring homomorphism $\mathbb{L} \xrightarrow{\varphi} B$ such that $\varphi(F_\Omega) = F_B^\gamma$, and we get a morphism of FGLs which provides the needed operation by Theorem 6.8. \square

The above two results provide an effective tool in constructing multiplicative operations. We will use them below to construct *Integral Adams Operations* and T. tom Dieck - style Steenrod operations in Algebraic Cobordism.

Let us describe the morphisms of FGLs (and so, the multiplicative operations between the respective theories) in some situations.

For $r > 1$, denote: $d(r) := G.C.D.(\binom{r}{i}, 0 < i < r)$. Then

$$d(r) = \begin{cases} p, & \text{if } r = p^k, \text{ for some } k; \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 6.10 *Let $(\varphi_G, \gamma_G) : (A, F_A) \rightarrow (B, F_B)$ be a morphism of FGLs. Then either $b_0 \neq 0$, or the first non-zero coefficient b_{r-1} of γ_G satisfies: $d(r) \cdot b_{r-1} = 0$.*

Proof: Suppose, $b_0 = 0$, and b_{r-1} is the first non-zero coefficient of γ_G . From the equality:

$$\varphi_G(F_A)(\gamma_G(x), \gamma_G(y)) = \gamma_G(F_B(x, y)),$$

we get: $b_{r-1}x^r + b_{r-1}y^r + \text{higher terms} = b_{r-1}(x+y)^r + \text{higher terms}$, which implies that $d(r) \cdot b_{r-1} = 0$. \square

Suppose, now B is an integral domain. Then the characteristic $\text{char}(B)$ is either a prime p , or 0.

1) $\text{char}(B) = 0$:

Corollary 6.11 *Let A^* and B^* be any theories in the sense of Definition 2.1 with torsion-free B , and $A^* \xrightarrow{G} B^*$ be a multiplicative operation. Then either $\gamma_G = 0$, or $b_0 \neq 0$.* \square

We will call operations with $b_0 \neq 0$ - *operations of the main type*. The respective γ will also be called *of the main type*.

2) $\text{char}(B) = p$:

Let B^* be a theory, where B is a ring of characteristic p . We can obtain a new theory $\text{Fr}(B)^*$ from B^* by the change of coefficients: $B \xrightarrow{Fr} B$. In particular, $F_{\text{Fr}(B)} = Fr(F_B)$. We have natural multiplicative operation: $\tilde{\text{Fr}} : \text{Fr}(B)^* \rightarrow B^*$ defined by: $\tilde{\text{Fr}}(u \otimes b) = u^p \cdot b$. The respective morphism of formal group laws will be: (id, x^p) .

Let $A^* \xrightarrow{G} B^*$ be a multiplicative operation, and $(\varphi, \gamma) : (A, F_A) \rightarrow (B, F_B)$ be a morphism of FGLs, such that $\gamma = b_{r-1}x^r + \dots$, and $b_{r-1} \neq 0$. Then it follows from Lemma 6.10, that $r = p^k$, for some $k \geq 0$.

Lemma 6.12 *In the above situation, $\gamma(x) = \delta(x^{p^k})$, for some $\delta \in B[[y]]$ with $\delta_0 \neq 0$.*

Proof: We need to show that degrees of all non-zero terms of γ are divisible by p^k . From the contrary, let b_{s-1} be the smallest coefficient with $p^k \nmid s$. Then looking at the degree s component of the equality: $\varphi(F_A)(\gamma(x), \gamma(y)) = \gamma(F_B(x, y))$, we get: $b_{s-1}x^s + b_{s-1}y^s = b_{s-1}(x+y)^s$, which implies that $d(s) \cdot b_{s-1} = 0$. But since $p \nmid s$, and B has characteristic p , this implies that $b_{s-1} = 0$. \square

Thus, any such morphism (φ, γ) of FGLs can be presented as the composition

$$(A, F_A) \xrightarrow{(\varphi, \delta)} (B, F_{\text{Fr}^k(B)}) \xrightarrow{(id, x^{p^k})} (B, F_B).$$

Return now to the situation where A^* is a theory of rational type. Then the morphism (φ, δ) of formal group laws defines a multiplicative operation $A^* \xrightarrow{H} \text{Fr}^k(B)^*$, and we get that $G = \tilde{\text{Fr}}^k \circ H$, where H is an operation of the main type.

Combining Theorem 6.9 with the above considerations, we get:

Theorem 6.13 *Let B^* be any theory in the sense of Definition 2.1 with B - an integral domain. Then:*

- 1) *If $\text{char}(B) = 0$, then the assignment $G \mapsto \gamma_G$ provides a 1-to-1 correspondence between multiplicative operations $\Omega^* \xrightarrow{G} B^*$ and such $\gamma = b_0x + \dots \in B[[x]]$ that either $\gamma = 0$, or $b_0 \neq 0$ and $F_B^\gamma \in B[b_0^{-1}][[x, y]]$ has coefficients in B .*
- 2) *If $\text{char}(B) = p$, then the assignment $G \mapsto (k, \gamma_H)$, where $G = \tilde{\text{Fr}}^k \circ H$, with H of the main type, provides a 1-to-1 correspondence between multiplicative operations $\Omega^* \xrightarrow{G} B^*$ and the pairs (k, γ) , where either $(k, \gamma) = (\infty, 0)$, or $k \in \mathbb{Z}_{\geq 0}$, and $\gamma = b_0x + \dots \in B[[x]]$ has $b_0 \neq 0$, and $(\text{Fr}^k(F_B))^\gamma \in B[b_0^{-1}][[x, y]]$ has coefficients in B .*

One can compose the morphisms of FGLs. Moreover, if (φ, γ) and (φ, β) have common homomorphism of coefficient rings, we can also "add" such morphisms of FGLs (just as one can add morphisms into an abelian group). Namely, we can set: $(\varphi, \beta) + (\varphi, \gamma) = (\varphi, \delta)$, where $\delta(x) = \varphi(F_A)(\beta(x), \gamma(x))$.

In particular, if A^* is a theory of rational type, and there exists only one endomorphism $A \xrightarrow{\varphi} A$, then the set of multiplicative operations $A^* \xrightarrow{G} A^*$ has a natural ring structure with multiplication = the composition, and addition as above. This happens for Chow groups, and for K_0 . In the case of CH^*/p , we get:

Theorem 6.14 *The ring of multiplicative operations $\text{CH}^*/p \rightarrow \text{CH}^*/p$ is $\mathbb{Z}/p[[\tilde{\text{Fr}}]]$. In particular, the composition is commutative.*

Proof: Since there is only one ring homomorphism $\mathbb{Z}/p \rightarrow \mathbb{Z}/p$, the multiplicative operations $\text{CH}^*/p \rightarrow \text{CH}^*/p$ are in 1-to-1 correspondence with the additive power series $\gamma(x) = \sum_r b_{p^r-1} x^{p^r}$. Moreover, $\text{Fr}(\text{CH}^*/p)^* = \text{CH}^*/p$, and $\tilde{\text{Fr}} : \text{CH}^*/p \rightarrow \text{CH}^*/p$ is given by the power series x^p . The composition of operations corresponds to the composition of γ 's, and addition is the usual addition of γ 's. Thus, our ring can be naturally identified with $\mathbb{Z}/p[[\tilde{\text{Fr}}]]$. \square

Under the identification above, the total Steenrod operation $\text{St}^{\text{Tot}} = id + S^1 + S^2 + \dots$ corresponds to $1 + \tilde{\text{Fr}}$, and the Integral Adams Operation Ψ_k (see below) corresponds to k . In particular, Ψ_0 which is identity on CH^0 and zero on CH^i , $i > 0$ corresponds to 0.

6.3 Integral Adams Operations

Adams operations Ψ_k provide an important tool in studying K -groups. In topology, analogous operations were constructed by S.P. Novikov for complex-oriented cobordisms MU in [10]. This construction required inverting k , since Ψ_k were basically expressed in terms of Landweber-Novikov operations, and the respective formulas do have k -denominators. Only much later it was shown by W.S. Wilson that these operations can be defined integrally and are naturally multiplicative unstable operations - see [24, Theorem 11.53]. Using our main results we can construct similar operations in Algebraic Cobordism and all other theories of rational type (it is worth noting, that although we produce a similar object, our methods have nothing to do with the methods of Wilson).

Theorem 6.15 *For any theory of rational type A^* , there are multiplicative (unstable) A -linear operations $\Psi_k : A^* \rightarrow A^*$, $k \in \mathbb{Z}$, such that $\gamma_{\Psi_k} = [k] \cdot_A x$.*

In the case of K_0 these are usual Adams operations.

Proof: Consider $\gamma_k = [k] \cdot_A x$. Since (id, γ_k) , is an endomorphism of FGL (A, F_A) , by Theorem 6.8, we get a multiplicative operation $\Psi_k : A^* \rightarrow A^*$ with such γ . \square

As the above operations are A -linear they can be obtained from the ones in Algebraic Cobordism by change of coefficients.

The set of Adams operations has common homomorphism of coefficient rings equal to the identity, and so form a ring $R_{\Psi, A}$. Clearly, Ψ_k is just the image of k under the canonical surjective ring homomorphism $\mathbb{Z} \rightarrow R_{\Psi, A}$. The operation Ψ_0 can be described as follows: it acts as id on constant elements, and as zero on \overline{A}^* . Thus, it is responsible for the decomposition which we used throughout the paper.

Adams operations can be used in the study of the graded Algebraic Cobordism (see [8, Subsection 4.5.2]). Being operations, they respect the codimension of support of an element: $\Psi_k(F^{(n)}\Omega^*(X)) \subset F^{(n)}\Omega^*(X)$, and so act on the graded ring $Gr^*\Omega^*(X)$. We have the natural surjection:

$$CH^* \otimes_{\mathbb{Z}} \mathbb{L}^* \twoheadrightarrow Gr^*\Omega^*,$$

which commutes with the action of Ψ_k (recall, that these operations are \mathbb{L} -linear). Thus, $\Psi_k|_{Gr^n\Omega^*}$ is the multiplication by k^n . Suppose now, $X \xrightarrow{f} Y$ is a morphism of smooth varieties. Then we get the morphism of the respective filtrations: $f^* : F^{(n)}\Omega^*(Y) \rightarrow F^{(n)}\Omega^*(X)$. This provides the spectral sequence computing Ker and Coker of f^* :

$$E_r^{p,q,n} \Rightarrow H^p(f^* : \Omega^q(Y) \rightarrow \Omega^q(X)),$$

where $E_2^{p,q,n} = H^p(Gr(f)^* : Gr^n\Omega^q(Y) \rightarrow Gr^n\Omega^q(X))$, $p = 0, 1$, and $d_r : E_r^{0,q,n} \rightarrow E_r^{1,q,n+r-1}$. Adams operations permit to estimate the exponent of d_r . Denote: $e(n, r) = \text{G.C.D.}(k^n(k^{r-1} - 1), k \in \mathbb{Z})$.

Proposition 6.16 $e(n, r) \cdot d_r|_{E_r^{0,q,n}} = 0$.

Proof: Since the Adams operation respects the filtration, it acts on the spectral sequence. Then Ψ_k must act as multiplication by k^n on $E_r^{p,q,n}$. Since $d_r : E_r^{0,q,n} \rightarrow E_r^{1,q,n+r-1}$, we get that, for any k , $k^n(k^{r-1} - 1)$ multiplied by such d_r is zero. \square

It is easy to see that $e(0, r) = 1$, and $e(n, 2s) = 2$, for all $n, s \geq 1$. And prime factors of $e(n, r)$ are exactly those p for which $(p-1)|(r-1)$. In particular, these do not depend on n . But the powers of these primes do. Thus, the "unstable information" is concentrated in these powers.

In particular, the above considerations apply to the extension of fields morphism.

6.4 Symmetric Operations for all primes, and T.tom Dieck - style Steenrod operations

These topics represent the main content of the paper [21]. Here we just present briefly the main results and ideas. The construction of Symmetric Operations for all primes was the main motivation behind the current paper. For about 5 years the author tried to construct them, until he realized that it is about as simple as constructing all unstable operations in Algebraic Cobordism. But let me start with the Steenrod operations.

Steenrod operations provide an important structure on CH^*/p which permits to do more elaborate tricks with algebraic cycles than the usual addition and multiplication. Individual Steenrod operations can be organized into "larger" multiplicative operations. One of the possible approaches is to consider the multiplicative operation: $St : \mathrm{CH}^*/p \rightarrow \mathrm{CH}^*/p[[t]]$ given by the morphism of FGLs (see Theorem 6.8): (φ, γ) , where $\varphi : \mathbb{Z}/p \rightarrow \mathbb{Z}/p[[t]]$ is the natural embedding (the unique morphism of rings), and $\gamma = -t^{p-1}x + x^p$ (notice, that our γ is additive in x). Then the individual Steenrod operation $S^r|_{\mathrm{CH}^m/p}$ will be the coefficient of $S|_{\mathrm{CH}^m/p}$ at $t^{(m-r)(p-1)}$. At the first glance it looks like we complicate things by making our operation unstable (the coefficient at x is not 1), but it appears to be convenient in various respects.

The original approach to Steenrod operations in Chow groups due to P.Brosnan (see [3]) is through \mathbb{Z}/p -equivariant Chow groups. In this construction, one produces the multiplicative operation $Sq : \mathrm{CH}^*(X)/p \rightarrow \mathrm{CH}^*(X)/p \otimes_{\mathbb{Z}/p} \mathrm{CH}^*(B\mathbb{Z}/p)/p$. We have $\mathrm{CH}^*(B\mathbb{Z}/p)/p = \mathbb{Z}/p[[t]]$, and one can show (see [3]) that the only non-trivial coefficients of Sq will be at $t^{r(p-1)}$, $r \geq 0$. The fact that the two constructions agree follows from Theorem 6.8 (the morphism of FGLs for Sq is easy to compute).

All of the above was known in topology for quite a while. And both mentioned constructions were extended to complex-oriented cobordism MU . The equivariant version is due to T. tom Dieck ([16]), and it goes completely parallel to the H^*/p (and CH^*/p) case. Here $MU^*(X \times B\mathbb{Z}/p) = MU^*(X)[[t]]/([p] \cdot MU(t))$, and one gets a multiplicative operation

$$Sq : MU^*(X) \rightarrow MU^*(X \times B\mathbb{Z}/p) \rightarrow MU^*(X)[[t]]/(\frac{[p] \cdot MU(t)}{t}).$$

The other version is due to D.Quillen ([14]). One observes that $-t^{p-1}x + x^p \equiv \prod_{i=0}^{p-1}(x + it) \pmod{p}$. Now we can produce an MU -analogue of this power series: $\gamma = \prod_{i=0}^{p-1}(x + MU[i] \cdot MU(t)) \in \mathbb{L}[[t]][[x]]$, which by universality of MU^* defines the multiplicative operation:

$$St : MU^* \rightarrow MU^*[[t]][t^{-1}][(p-1)!^{-1}].$$

Notice, that this time, we have to invert t and $(p-1)!$, since the shifted formal group law $F_{MU[[t]]}^\gamma$ has denominators. Also, St has non-trivial coefficients at t^j , for j not divisible by $(p-1)$. It was shown by D.Quillen that his approach agrees with the one of T. Tom Dieck. More precisely, one has the following commutative diagram:

$$\begin{array}{ccc} MU^* & \xrightarrow{St} & MU^*[[t]][t^{-1}][(p-1)!^{-1}] \\ Sq \downarrow & & \downarrow \\ MU^*[[t]]/(\frac{[p] \cdot MU(t)}{t}) & \longrightarrow & MU^*[[t]][t^{-1}]/([p] \cdot MU(t)). \end{array}$$

Let us try to extend these constructions to the case of Algebraic Cobordism Ω^* . The Quillen's version is completely straightforward. Here one needs only the universality of Ω^* supplied by M.Levine-F.Morel ([8, Theorem 1.2.6]) and the change of orientation of I.Panin-A.Smirnov ([12]). Let us do a more general case (suggested by D.Quillen). Namely, chose representatives $\{i_j, 0 < j < p\}$ of all non-zero cosets modulo

p , and denote $\mathbf{i} := \prod_{j=1}^{p-1} i_j$. Then we can consider the power series $\gamma = \prod_{j=0}^{p-1} (x + {}_\Omega [i_j] \cdot {}_\Omega t) \in \mathbb{L}[[t]][[x]]$, which, by Theorem 3.8, defines the multiplicative operation

$$St(\bar{i}) : \Omega^* \rightarrow \Omega^*[[t]][t^{-1}][\mathbf{i}^{-1}].$$

The situation with the version of Tom Dieck is rather different. Although one can easily define the \mathbb{Z}/p -equivariant Algebraic Cobordism $\Omega_{\mathbb{Z}/p}^*(X)$, one encounters problems trying to prove that the natural map $\Omega^n(X) \rightarrow \Omega_{\mathbb{Z}/p}^{np}(X^{\times p})$ is well-defined. It is easy to show that the standard cobordism relations are respected, but the author was unable to handle the double-points relations. The only case where the author succeeded was $p = 2$, where he had to employ the Symmetric Operations (modulo 2) constructed in [17], [19]. These operations, which are more subtle than the Steenrod ones, until now were unavailable for $p > 2$.

Hopefully, our Theorem 6.8 permits to construct what we need.

Theorem 6.17 ([21]) *There is the multiplicative operation Sq which fits into the commutative diagram:*

$$\begin{array}{ccc} \Omega^* & \xrightarrow{St(\bar{i})} & \Omega^*[[t]][t^{-1}][\mathbf{i}^{-1}] \\ Sq \downarrow & & \downarrow \\ \Omega^*[[t]]/(\frac{[p] \cdot {}_\Omega(t)}{t}) & \longrightarrow & \Omega^*[[t]][t^{-1}]/([p] \cdot {}_\Omega(t)). \end{array}$$

Notice, that Sq is a bit more "canonical" than St - it does not depend on \bar{i} .

Now, since the target of Sq has no negative powers of t , the commutativity of the above diagram shows that the *negative part* of $St(\bar{i})$ is divisible by $\frac{[p] \cdot {}_\Omega(t)}{t}$. I should point out that this fact itself does not require the above Theorem, or the methods of the current paper. But what is much deeper, it appears that one can divide "canonically", and the quotient is what we call *Symmetric operation*.

Theorem 6.18 ([21]) *There is an operation $\Phi(\bar{i}) : \Omega^* \rightarrow \Omega^*[t^{-1}][\mathbf{i}^{-1}]$ such that*

$$(St(\bar{i}) - \frac{[p] \cdot {}_\Omega(t)}{t} \cdot \Phi(\bar{i})) : \Omega^* \rightarrow \Omega^*[[t]][\mathbf{i}^{-1}].$$

Some traces of the MU -analogue of this operation were used by D.Quillen in [14], and they provide the main tool of the mentioned article.

In Algebraic Cobordism the described operation appeared originally in the works [17] and [19] of the author in the case $p = 2$ in a different form. Namely, in the form of "slices", which were constructed geometrically. Only substantially later the author had realized that these slices can be combined into the "formal half" of the "negative part" of some multiplicative operation, which had a power series $\gamma = x \cdot (x + {}_\Omega t)$ reminiscent of a Steenrod operation in Chow groups mod 2. How to view the operation $\Phi(\bar{i})$? The natural approach would be to consider the coefficients of it at particular monomials t^{-n} , or, in other words, $\text{Res}_{t=0} \frac{t^n \cdot \Phi(\bar{i}) \omega_t}{t}$. And, if one thinks about it, there is no point restricting yourself to monomials, so one can consider

$$\Phi(\bar{i})^{q(t)} := \text{Res}_{t=0} \frac{q(t) \cdot \Phi(\bar{i}) \omega_t}{t},$$

where $q(t) = q_1 t + q_2 t^2 + \dots \in \mathbb{L}[[t]]$ is any power series without the constant term. Of course, there are various relations among these slices which bind them together into something "larger" - the operation $\Phi(\bar{i})$. For $p = 2$, these are exactly the Symmetric operations $\Phi^{q(t)}$ of [19]:

Proposition 6.19 ([21]) *In the case $p = 2$, with $\bar{i} = \{-1\}$, for any power series as above, we have:*

$$\Phi(\bar{i})^{q(t)} = -\Phi^{q(t)}.$$

Notice, that for $p = 2$, there is, in addition, a non-additive operation Φ^1 (see [19]). At the moment, we can not produce it's analogues for $p > 2$ as our methods so far are restricted to additive operations only. Hopefully, additive Symmetric Operations are sufficient for most applications.

The cases $p = 2$ and 3 are special, since we can choose our representatives \bar{i} to be invertible in \mathbb{Z} . For $p = 2$, we have two such choices: $\{1\}$, or $\{-1\}$ (in [19], $\{-1\}$ was "chosen"). For $p = 3$, the choice is canonical: $\{1, -1\}$. Thus, we get integral operations $\Phi(\bar{i}) : \Omega^* \rightarrow \Omega^*[t^{-1}]$. And, for arbitrary p , we can choose our remainders to be the powers of some fixed prime l (generating $(\mathbb{Z}/p)^*$), so that only one prime would be inverted. Moreover, this prime can be chosen in infinitely many ways, so, in a sense, the picture is as good as integral.

For $p = 2$ the Symmetric operations were applied to the study of 2-torsion effects in Chow groups - they provide the only known method to get "clean results" on rationality - see [18] and [20]. And similar applications are expected for other primes. Other applications involve the study of the structure of the \mathbb{L} -module $Gr\Omega^*(X)$. Here the construction of Symmetric Operations for all primes changes the statements $\otimes \mathbb{Z}_{(2)}$ into integral ones.

7 Basic tools

Here we present various results which permit to work effectively with cohomology theories.

7.1 Projective bundle and blow-up results

We start with the *excess intersection formula* - see [19, Theorem 5.19] and [8, Theorem 6.6.9]. Consider cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f'} & Z \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

with f, f' - regular embeddings, and $(g')^*(\mathcal{N}_{Y \subset X})/\mathcal{N}_{W \subset Z} =: \mathcal{M}$ the vector bundle of dimension d .

Proposition 7.1 *Let A^* be any theory in the sense of Definition 2.1. In the above situation,*

$$g^* f_*(v) = f'_*(c_d^A(\mathcal{M}) \cdot (g')^*(v));$$

If g is projective, then also:

$$f^* g_*(u) = g'_*(c_d^A(\mathcal{M}) \cdot (f')^*(u)).$$

Proof: Both of the above references are dealing with the Ω^* -case. Although, the statement of [8, Theorem 6.6.9] is more general, it requires the development of the whole theory of refined pull-backs. For Algebraic Cobordism such a theory is constructed in [8], but it requires some work to extend it to a more general context. In contrast, the proof of [19, Theorem 5.19] does not use any specifics of Ω^* and works in general.

□

Another important tool is the formula of Quillen - see [13, Theorem 1], [12, Formula (24)], and [19, Theorem 5.35]. It describes push-forwards for projective bundles.

Recall that, for an n -dimensional vector bundle \mathcal{V} , the *roots* are elements $\lambda_i \in A^1(Flag_X(\mathcal{V}))$, $i = 1, \dots, n$ such that $\prod_{i=1}^n (t + \lambda_i) = \sum_{i=0}^n c_i^A(\mathcal{V}) t^{n-i}$, where $Flag_X(\mathcal{V})$ is a variety of complete flags of \mathcal{V} , and $c_i^A(\mathcal{V})$ are Chern classes in the theory A^* . The important point here is that the pull-back map $A^*(X) \rightarrow A^*(Flag_X(\mathcal{V}))$ is split injective.

Recall also that $\omega_A \in A[[x]]dx$ is the canonical invariant 1-form satisfying: $w_A(0) = dx$. Such a form can be obtained from the formal group law $F_A(x, y)$ of A^* by the formula: $\omega_A = \left(\frac{\partial F_A}{\partial y} \Big|_{y=0} \right)^{-1} dx$. By the formula of Mistchenko it can be expressed as:

$$([\mathbb{P}^0]_A + [\mathbb{P}^1]_A \cdot x + [\mathbb{P}^2]_A \cdot x^2 + \dots) dx,$$

where $[\mathbb{P}^r]_A$ is the class of \mathbb{P}^r in $A^*(\text{Spec}(k)) = A$.

Proposition 7.2 *Let A^* be any theory in the sense of Definition 2.1. Let X be some smooth quasi-projective variety, \mathcal{V} be some n -dimensional vector bundle on it, and $\pi : \mathbb{P}_X(\mathcal{V}) \rightarrow X$ be the corresponding projective bundle. Let $f(t) \in A^*(X)[[t]]$, and $\xi = c_1^A(\mathcal{O}(1))$. Then*

$$\pi_*(f(\xi)) = \text{Res}_{t=0} \frac{f(t) \cdot \omega_A}{\prod_i (t +_A \lambda_i)},$$

where λ_i are roots of \mathcal{V} , and $+_A$ is the formal addition in the sense of F_A .

Proof: Clearly, both parts of the formula are $A^*(X)$ -linear, so it is sufficient to prove the result in the case: $f(t) = t^r$ - a monomial. Then it formally follows from the Ω^* -case proven in [19, Theorem 5.35] (using the universality of Ω^* - [8, Theorem 1.2.6]). \square

We will need various results concerning the blow up morphism.

Let X be smooth variety, R it's smooth closed subvariety, $\tilde{X} = Bl_{X,R}$ - the blow up of X at R , and E - the exceptional divisor on \tilde{X} . These fit into the blow-up diagram:

$$\begin{array}{ccc} E & \xrightarrow{j} & \tilde{X} \\ \varepsilon \downarrow & & \downarrow \pi \\ R & \xrightarrow{i} & X. \end{array}$$

Let \mathcal{N} be the normal bundle of R in X , then $E \cong \mathbb{P}_R(\mathcal{N})$. Let $d = \dim(\mathcal{N}) = \text{codim}(R \subset X)$. Denote $\tilde{\mathcal{N}} = \mathcal{N} \oplus \mathcal{O}(1)$, and $\tilde{E} = \mathbb{P}_R(\tilde{\mathcal{N}}) \xrightarrow{\tilde{\varepsilon}} R$.

The following result of M. Levine and F. Morel describes the class of the blow up in the A^* of the base.

Proposition 7.3 ([8, Proposition 2.5.2]) *Let A^* be any theory in the sense of Definition 2.1. Then*

$$\pi_*(1) = 1 + i_* \tilde{\varepsilon}_* \left(\frac{c_1^A(\mathcal{O}(1))}{c_1^A(\mathcal{O}(-1))} \right).$$

In particular, the above class is invertible. More generally, we have:

Proposition 7.4 *Let A^* be any theory in the sense of Definition 2.1, and $\pi : \tilde{X} \rightarrow X$ be projective bi-rational morphism of smooth varieties. Then*

- (1) $\pi_*(1)$ is invertible in $A^*(X)$.
- (2) $\pi_* : A_*(\tilde{X}) \rightarrow A_*(X)$ is surjective.

Proof: By universality of Ω^* ([8, Theorem 1.2.6]), we have the canonical map of theories $\Omega^* \rightarrow A^*$, and $\pi_*(1)$ is in the image of this map. So, it is sufficient to treat the case $A^* = \Omega^*$. Since π is bi-rational, we have a closed subscheme $Z \subset X$ of positive codimension, such that π is an isomorphism outside Z . Then $\pi_*(1) = 1 + u$, where u is supported on Z . That means that u has positive codimension of support, and so is nilpotent by [19, Statement 5.2]. Hence, $\pi_*(1)$ is invertible. It remains to apply the projection formula. \square

The following result describes what happens to the whole A^* when you blow up some smooth variety at a smooth center.

Proposition 7.5 (cf.[19, Proposition 5.24]) *Let A^* be any theory in the sense of Definition 2.1. Then we have split exact sequences:*

$$(1) \quad 0 \longleftarrow A_*(X) \xleftarrow{\pi^*, -i_*} A_*(\tilde{X}) \oplus A_*(R) \xleftarrow{j^*, \varepsilon_*} A_*(E) \longleftarrow 0;$$

$$(2) \quad 0 \longrightarrow A^*(X) \xrightarrow{\pi^*, -i^*} A^*(\tilde{X}) \oplus A^*(R) \xrightarrow{j^*, \varepsilon^*} A^*(E) \longrightarrow 0.$$

Proof: In the case $A^* = \Omega^*$, (1) was proven in [19, Proposition 5.24], and the same proof works for arbitrary A^* . Let us recall some details. Let $\mathcal{K} = \varepsilon^* \mathcal{N} / \mathcal{O}(-1)$ be the excess bundle on E . It is easy to see (see [19, Proposition 5.22]) that the class of the diagonal on $E \times_R E$ is given by $c_{d-1}^A(\mathcal{K}_1 \otimes \mathcal{O}(1)_2)$, where \mathcal{V}_l denotes the bundle \mathcal{V} lifted from the l -th component. This class can be written as $c_{d-1}^A(\mathcal{K}) \times 1 + \sum_{i \geq 1} \gamma_{d-1-i} \times \zeta^i$, where $\gamma_j \in A^j(E)$ are some elements, and $\zeta = c_1^A(\mathcal{O}(-1))$. Let us introduce the elements $\alpha := c_{d-1}^A(\mathcal{K}) \in A^*(E)$, and $\beta := \tilde{\varepsilon}_* \left(\frac{c_1^A(\mathcal{O}(1))}{c_1^A(\mathcal{O}(-1))} \right) \in A^*(R)$. Then for any $u \in A^*(E)$, we have:

$$u = \alpha \cdot \varepsilon^* \varepsilon_*(u) + \sum_{j \geq 1} \gamma_{d-1-j} \cdot \varepsilon^* \varepsilon_*(u \cdot \zeta^j);$$

$$u = \varepsilon^* \varepsilon_*(u \cdot \alpha) + \sum_{j \geq 1} \zeta^j \cdot \varepsilon^* \varepsilon_*(u \cdot \gamma_{d-1-j}),$$

Consider the maps $F : A_*(E) \rightarrow A_*(E)$ and $G : A^*(E) \rightarrow A^*(E)$ given by:

$$F(u) = \sum_{j \geq 0} \gamma_{d-2-j} \cdot \varepsilon^* \varepsilon_*(u \cdot \zeta^j); \quad G(u) = \sum_{j \geq 0} \zeta^j \cdot \varepsilon^* \varepsilon_*(u \cdot \gamma_{d-2-j}).$$

Consider the diagram:

$$\begin{array}{ccccc} E \times_R E & \xrightarrow{id \times e} & E \times_R \tilde{E} & \xrightarrow{\rho} & \tilde{E} \\ & \searrow p_1 & \downarrow \tilde{\rho} & & \downarrow \tilde{\varepsilon} \\ & & E & \xrightarrow{\varepsilon} & R. \end{array}$$

Let $E \xrightarrow{e} \tilde{E}$ be the natural embedding. Then $e_*(1) = c_1^A(\mathcal{O}(1))$. We get:

$$\begin{aligned}
F(1) &= (p_1)_* \left(\frac{c_{d-1}^A(\mathcal{K}_1 \otimes \mathcal{O}(1)_2) - c_{d-1}^A(\mathcal{K}_1)}{c_1^A(\mathcal{O}(-1)_2)} \right) \\
&= \tilde{\rho}_* \left((c_{d-1}^A(\mathcal{K}_1 \otimes \mathcal{O}(1)_2) - c_{d-1}^A(\mathcal{K}_1)) \cdot \frac{c_1^A(\mathcal{O}(1)_2)}{c_1^A(\mathcal{O}(-1)_2)} \right) \\
&= \tilde{\rho}_* \left(c_{d-1}^A(\mathcal{K}_1 \otimes \mathcal{O}(1)_2) \frac{c_1^A(\mathcal{O}(1)_2)}{c_1^A(\mathcal{O}(-1)_2)} \right) - c_{d-1}^A(\mathcal{K}) \cdot \tilde{\rho}_* \rho^* \left(\frac{c_1^A(\mathcal{O}(1))}{c_1^A(\mathcal{O}(-1))} \right) \\
&= \operatorname{Res}_{t=0} \frac{c_{\bullet}^A(\mathcal{K})(t) \cdot t \cdot \omega_A}{c_{\bullet}^A(\tilde{\mathcal{N}})(t) \cdot (-At)} - c_{d-1}^A(\mathcal{K}) \cdot \varepsilon^* \tilde{\varepsilon}_* \left(\frac{c_1^A(\mathcal{O}(1))}{c_1^A(\mathcal{O}(-1))} \right) \\
&= \operatorname{Res}_{t=0} \frac{\omega_A}{(t + {}_A\xi)(-At)} - \alpha \cdot \varepsilon^*(\beta) = -\alpha \cdot \varepsilon^*(\beta).
\end{aligned}$$

Now we can construct contracting homotopies λ and μ for (1) and (2):

$$\begin{array}{ccc}
A_*(E) & \xrightarrow{d_2} & A_*(\tilde{X}) \\
\lambda_1 \uparrow & \lambda_2 & \lambda_4 \uparrow \\
A_*(R) & \xrightarrow{d_3} & A_*(X), \\
& \lambda_3 &
\end{array}
\quad
\begin{array}{ccc}
A^*(E) & \xrightarrow{\mu_2} & A^*(\tilde{X}) \\
\mu_1 \uparrow & \mu_2 & \mu_4 \uparrow \\
A^*(R) & \xrightarrow{\mu_3} & A^*(X), \\
& \mu_3 &
\end{array}$$

in the following way: $\lambda_4 = \pi^*$; $\lambda_3 = \beta \cdot i^*$, $\lambda_1 = \alpha \cdot \varepsilon^*$; and $\lambda_2 = F \circ j^*$, while $\mu_4 = \pi_*$; $\mu_3 = i_*(\beta \cdot \cdot)$, $\mu_1 = \varepsilon_*(\alpha \cdot \cdot)$; and $\mu_2 = j_* \circ G$.

From the equality $F(1) = -\alpha \cdot \varepsilon^*(\beta)$ (using several times the projection formula) one easily obtains the left ones of the following identities:

$$\lambda_2 \circ \lambda_4 = -\lambda_1 \circ \lambda_3; \quad d_2 \circ \lambda_1 = -\lambda_4 \circ d_3; \quad (5)$$

$$\mu_4 \circ \mu_2 = -\mu_3 \circ \mu_1; \quad \mu_1 \circ d_2 = -d_3 \circ \mu_4, \quad (6)$$

while the right ones are the Excess Intersection Formula (Proposition 7.1). The identity: $d_3 \circ \lambda_3 + d_4 \circ \lambda_4 = id_{A_*(X)}$ is just the Proposition 7.3 (plus the projection formula). The identity: $d_1 \circ \lambda_1 + \lambda_3 \circ d_3 = id_{A_*(R)}$ follows from the Excess Intersection Formula and Proposition 7.3. The identity: $\lambda_1 \circ d_1 + \lambda_2 \circ d_2 = id_{A_*(E)}$ follows from the definition of F . Finally, the identity: $d_2 \circ \lambda_2 + \lambda_4 \circ d_4 = id_{A_*(\tilde{X})}$ follows from the ones already proven, plus (5), plus the fact that the map $(j_*, \pi^*) : A_*(E) \oplus A_*(X) \rightarrow A_*(\tilde{X})$ is surjective, which follows from the $(EXCI)$ axiom (see the proof of [19, Proposition 5.24]).

The identity: $\mu_3 \circ d_3 + \mu_4 \circ d_4 = id_{A^*(X)}$ follows from Proposition 7.3, and the projection formula. The identity: $\mu_1 \circ d_1 + d_2 \circ \mu_2 = id_{A^*(R)}$ follows from the Excess Intersection Formula and Proposition 7.3. The identity: $d_1 \circ \mu_1 + d_2 \circ \mu_2 = id_{A^*(E)}$ follows from the definition of G . Finally, the identity: $\mu_2 \circ d_2 + d_4 \circ \mu_4 = id_{A^*(\tilde{X})}$ follows the ones already proven, plus (6), plus the fact that the map $(j^*, \pi_*) : A^*(\tilde{X}) \rightarrow A^*(E) \oplus A^*(X)$ is injective, which follows from the fact that λ is a contracting homotopy for the complex (1). \square

Proposition 7.6 *Let A^* be any generalized oriented cohomology theory in the sense of Definition 2.1, and $\pi : \tilde{V} \rightarrow V$ be the permitted blow up of a smooth variety with smooth centers R_i and the respective components of the exceptional divisor $E_i \xrightarrow{\varepsilon_i} R_i$. Then one has exact sequences:*

$$(1) \quad 0 \leftarrow A_*(V) \xleftarrow{\pi_*} A_*(\tilde{V}) \leftarrow \bigoplus_i \operatorname{Ker}(A_*(E_i) \xrightarrow{(\varepsilon_i)_*} A_*(R_i)).$$

$$(2) \quad 0 \rightarrow A^*(V) \xrightarrow{\pi^*} A^*(\tilde{V}) \rightarrow \oplus_i \text{Coker}(A^*(R_i) \xrightarrow{(\varepsilon_i)^*} A^*(E_i))$$

Proof: The Proposition 7.5 settles the case where π is a single blow up. Let us use induction on the number of blowings. Suppose, \tilde{V} is the result of n blowings, and Y is the result of $(n-1)$ (first) of them. Then $\rho : \tilde{V} \rightarrow Y$ is a single blow up with the center R . Let $F_i, i = 1, \dots, n-1$ be the components of the exceptional divisor of Y , and $E_i, i = 1, \dots, n-1$ be there proper preimages under ρ , and E be the exceptional divisor of ρ . By inductive assumption and Proposition 7.5, we have exact sequences:

$$\begin{aligned} 0 \leftarrow A_*(V) &\xleftarrow{\pi_*} A_*(Y) \leftarrow \oplus_{i=1}^{n-1} \text{Ker}(A_*(F_i) \xrightarrow{(\varepsilon_i)^*} A_*(R_i)); \\ 0 \rightarrow A^*(V) &\xrightarrow{\pi^*} A^*(Y) \rightarrow \oplus_{i=1}^{n-1} \text{Coker}(A^*(R_i) \xrightarrow{(\varepsilon_i)^*} A^*(F_i)); \\ 0 \leftarrow A_*(Y) &\xleftarrow{\rho_*} A_*(\tilde{V}) \leftarrow \text{Ker}(A_*(E) \xrightarrow{\varepsilon_*} A_*(R)); \\ 0 \rightarrow A^*(Y) &\xrightarrow{\rho^*} A^*(\tilde{V}) \rightarrow \text{Coker}(A^*(R) \xrightarrow{\varepsilon^*} A^*(E)). \end{aligned}$$

Taking into account that the maps:

$$\begin{aligned} \text{Ker}(A_*(F_i) \rightarrow A_*(R_i)) &\leftarrow \text{Ker}(A_*(E_i) \rightarrow A_*(R_i)); \\ \text{Coker}(A^*(R_i) \rightarrow A^*(F_i)) &\hookrightarrow \text{Coker}(A^*(R_i) \rightarrow A^*(E_i)) \end{aligned}$$

are surjective and injective, respectively, we get the needed exact sequences:

$$\begin{aligned} 0 \leftarrow A_*(V) &\leftarrow A_*(\tilde{V}) \leftarrow \text{Ker}(A_*(E) \rightarrow A_*(R)) \oplus \left(\oplus_{i=1}^{n-1} \text{Ker}(A_*(E_i) \rightarrow A_*(R_i)) \right); \\ 0 \rightarrow A^*(V) &\rightarrow A^*(\tilde{V}) \rightarrow \text{Coker}(A^*(R) \rightarrow A^*(E)) \oplus \left(\oplus_{i=1}^{n-1} \text{Coker}(A^*(R_i) \rightarrow A^*(E_i)) \right). \end{aligned}$$

□

The following "singular" variant of the above result permits to present $A_*(Z)$ in terms of A_* of finitely many smooth varieties.

Proposition 7.7 *Let Z be a variety, and $\tilde{Z} \xrightarrow{\pi} Z$ be the blow up with centers R_i and exceptional divisors E_i . Then we have an exact sequence:*

$$0 \leftarrow A_*(Z) \leftarrow \left(A_*(\tilde{Z}) \oplus (\oplus_i A_*(R_i)) \right) \leftarrow \oplus_i A_*(E_i).$$

Proof:

Lemma 7.8 *Let $\pi : \tilde{V} \rightarrow V$ be projective birational map of smooth varieties, which is an isomorphism outside the closed subvariety $T \rightarrow V$, and such that $W = \pi^{-1}(T)$ is a divisor with strict normal crossing with components E_i . Then we have an exact sequence:*

$$0 \leftarrow A_*(V) \xleftarrow{\pi_*} A_*(\tilde{V}) \leftarrow \oplus_i \text{Ker}(A_*(E_i) \rightarrow A_*(T)).$$

Proof: Let $\pi' : \tilde{V}' \rightarrow V$ be the permitted blow up with centers over T resolving T to a divisor W' with strict normal crossing (Theorem 8.4). Let E'_j be the components of W' , and R'_j be the respective smooth centers. Then, by Proposition 7.6, we have an exact sequence:

$$0 \leftarrow A_*(V) \xleftarrow{\pi'_*} A_*(\tilde{V}') \leftarrow \oplus_i \text{Ker}(A_*(E'_i) \rightarrow A_*(R'_i)).$$

Since the map $A_*(\tilde{V}')/(\oplus_i \text{Ker}(A_*(E'_i) \rightarrow A_*(R'_i))) \rightarrow A_*(V)$ clearly factors through $A_*(\tilde{V}')/(\oplus_i \text{Ker}(A_*(E'_i) \rightarrow A_*(T)))$, we have the statement for \tilde{V}' . Let us denote $B(\tilde{V}) := \text{Coker} \left(\oplus_i \text{Ker}(A_*(E_i) \rightarrow A_*(T)) \rightarrow A_*(\tilde{V}) \right)$.

We have a natural surjective map $B(\tilde{V}) \twoheadrightarrow A_*(V)$. Since \tilde{V} and \tilde{V}' are isomorphic outside W and W' , by the Weak Factorization Theorem (Theorem 8.6(6)), we have a diagram:

$$\begin{array}{ccccccc} & Y_1 & & Y_3 & & Y_{n-2} & & Y_n \\ & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow \\ \tilde{V}' & & Y_2 & & Y_4 & \dots & Y_{n-3} & & Y_{n-1} & & \tilde{V} \end{array}$$

where all Y_i 's are projective either over \tilde{V}' , or \tilde{V} , and all the maps are blowings up/down w.r.to smooth centers which belong to exceptional divisor, and meet all of it's components properly. In particular, each Y_i has a natural map to V , which is an isomorphism outside T , and the preimage of T is the exceptional divisor (with normal crossing) on Y_i . Since the maps $Y_{2n-1} \rightarrow Y_{2n} \leftarrow Y_{2n+1}$ are blowings up/down with centers belonging to an exceptional divisor, we see (using Proposition 7.6) that the maps $B(Y_{2n-1}) \rightarrow B(Y_{2n}) \leftarrow B(Y_{2n+1})$ are isomorphisms. Clearly, these identifications are compatible with the maps $B(Y_i) \rightarrow A_*(V)$. Since the map $B(\tilde{V}') \rightarrow A_*(V)$ is an isomorphism, so is the map $B(\tilde{V}) \rightarrow A_*(V)$. \square

Lemma 7.9 *Let $\tilde{Z} \xrightarrow{\pi} Z$ be a projective map of varieties, which is an isomorphism outside the closed subvariety $R \rightarrow Z$ with the preimage $E = \pi^{-1}(R)$. Then one has an exact sequence:*

$$0 \leftarrow A_*(Z) \leftarrow (A_*(\tilde{Z}) \oplus A_*(R)) \leftarrow A_*(E).$$

Proof: The fact that it is a complex is evident.

Let us construct the map

$$A_*(Z) \xrightarrow{\varphi} \text{Coker} \left(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R) \right)$$

inverse to our projection. Let $v : V \rightarrow Z$ be some projective map with V smooth irreducible. If image of v is contained in R , then we get a natural map $A_*(V) \rightarrow A_*(R) \rightarrow \text{Coker}(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R))$. Otherwise, we have a birational map $V \dashrightarrow \tilde{Z}$, which can be resolved by blowing smooth centers over $v^{-1}(R)$. Then the exceptional set on \tilde{V} is the divisor with strict normal crossing W , which is mapped to E via \tilde{v} . Moreover, we can assume that W coincides with the preimage of R . If F_j are components of W , and S_j are the respective smooth centers, then by Lemma 7.8,

$$0 \leftarrow A_*(V) \xleftarrow{\pi^*} A_*(\tilde{V}) \leftarrow \oplus_j \text{Ker}(A_*(F_j) \rightarrow A_*(v^{-1}(R))).$$

Since F_j are mapped to E , and $v^{-1}(R)$ to R , the map $(\tilde{v})_* : A_*(\tilde{V}) \rightarrow A_*(\tilde{Z})$ provides a well-defined map $A_*(V) \xrightarrow{\varphi} \text{Coker} \left(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R) \right)$.

Let \tilde{V}_1, \tilde{V}_2 be two resolutions as above, with the exceptional divisors W_1 and W_2 . Then $\tilde{V}_1 \setminus W_1 \cong V \setminus v^{-1}(R) \cong \tilde{V}_2 \setminus W_2$. Hence, by the Weak Factorization Theorem (Theorem 8.6(6)), there exists a diagram:

$$\begin{array}{ccccccc} & Y_1 & & Y_3 & & Y_{n-2} & & Y_n \\ & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow \\ \tilde{V}_1 & & Y_2 & & Y_4 & \dots & Y_{n-3} & & Y_{n-1} & & \tilde{V}_2 \end{array}$$

where all Y_i 's are projective either over \tilde{V}_1 , or \tilde{V}_2 , and all the maps are blowings up/down w.r.to smooth centers which belong to exceptional divisor, and meet all of it's components properly. In particular, each Y_i has a natural map to \tilde{Z} , so that the preimage of E is the exceptional divisor. Using notations from the proof of Lemma 7.8, define $B(Y_i) := \text{Coker}((\oplus_j \text{Ker}(A_*(G_j) \rightarrow A_*(v^{-1}(R)))) \rightarrow A_*(Y_i))$, where G_j

are components of the exceptional divisor of Y_i . Then we have a natural map $B(Y_i) \rightarrow \text{Coker}(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R))$, which is compatible with the identifications: $B(Y_{2n-1}) = B(Y_{2n}) = B(Y_{2n+1})$ (as in the proof of Lemma 7.8). This shows that the map $A_*(V) \xrightarrow{\varphi_v} \text{Coker}(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R))$ does not depend on the choice of the resolution $\tilde{V} \rightarrow V$.

Let $V_1 \xrightarrow{f} V_2 \xrightarrow{v_2} Z$ be some projective maps with V_1 and V_2 smooth, and $v_1 = v_2 \circ f$. We can assume V_1 and V_2 irreducible. If the $\text{image}(v_2) \subset R$, then both maps φ_{v_1} and φ_{v_2} are passing through $A_*(R)$ and are clearly compatible with f_* . So, we can assume that $\text{image}(v_2) \not\subset R$. Let $\tilde{V}_2 \rightarrow V_2$ be the permitted blow up resolving indeterminacy of $\pi^{-1} \circ v_2$, and resolving $v_2^{-1}(R)$ to a divisor with normal crossing W_2 .

If $\text{image}(v_1) \subset R$, then since the fibers of the projection $\tilde{V}_2 \rightarrow V_2$ are rational, we get a rational map $V_1 \dashrightarrow W_2$. Resolve the indeterminacies of this map: $V_1 \xleftarrow{\rho} \tilde{V}_1 \xrightarrow{f'} W_2$, which gives $\tilde{f} : \tilde{V}_1 \rightarrow \tilde{V}_2$. Since the map $\rho_* : A_*(\tilde{V}_1) \rightarrow A_*(V_1)$ is surjective, and the compatibility of this map with $\varphi_{\tilde{v}_1}, \varphi_{v_1}$ is already known (the image is in R), we can substitute V_1 by \tilde{V}_1 . Since W_2 is mapped to E , we get that $\varphi_{v_2} \circ \tilde{f}_* = \varphi_{\tilde{v}_1} : A_*(\tilde{V}_1) \rightarrow \text{Coker}(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R))$.

Finally, if $\text{image}(v_1) \not\subset R$, then we get a rational map $V_1 \dashrightarrow \tilde{V}_2$ with indeterminacies only over $v_1^{-1}(R)$ which can be resolved by $\tilde{V}_1 \rightarrow V_1$ making the preimage of R the divisor with normal crossing W_1 . We get a map $\tilde{f} : \tilde{V}_1 \rightarrow \tilde{V}_2$. Then we can take $\tilde{v}_1 = \tilde{v}_2 \circ \tilde{f}$, and so $\varphi_{v_1} = \varphi_{v_2} \circ f_*$.

Thus, we get a well-defined map:

$$A_*(Z) \xrightarrow{\varphi} \text{Coker}(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R))$$

It is easy to see that it is inverse to the natural projection:

$$A_*(Z) \xleftarrow{\psi} \text{Coker}(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R))$$

On the left: let $v : V \rightarrow Z$ be some projective map with V smooth irreducible. There are two cases: 1) $\text{image}(v) \subset R$; 2) $\text{image}(v) \not\subset R$. In both cases, the fact that $\psi \circ \varphi|_{A_*(V)}$ is the identity is evident from the very definition.

On the right: the fact that $\varphi \circ \psi$ is the identity on the $A_*(R)$ -component is evident. As for the $A_*(\tilde{Z})$ -component, if we have some projective map $v : V \rightarrow \tilde{Z}$ with the resolution $\rho : \tilde{V} \rightarrow V$ of the closed subvariety $v^{-1}(E)$, then the map $\varphi_v \circ \psi|_{A_*(V)}$ is just $v_* : A_*(V) \rightarrow \text{Coker}(A_*(E) \rightarrow A_*(\tilde{Z}) \oplus A_*(R))$ (since $v \circ \rho$ factors through v). Thus, we get the identity map on the $A_*(\tilde{Z})$ -component as well. Hence, our complex is exact. \square

The Lemma 7.9 settles the case where π is a single blow up. The rest is done by the induction on the number of blowings in the same way as the proof of Proposition 7.6. \square

Remark: 1) In particular, this applies when $\tilde{Z} \rightarrow Z$ is the resolution of Z as in Theorems 8.2, 8.3, that is the permitted blow up with smooth centers which meet the components of the exceptional divisor properly, and resolves the singularities of Z , and then makes the special divisor the one with the strict normal crossing. In this case, all the varieties aside from Z participating in the formula are smooth, and we get the "finite" presentation of $A_*(Z)$ in terms of smooth varieties.

2) The map $A_*(\tilde{Z}) \rightarrow A_*(Z)$ is not surjective, in general, if Z is not smooth. Take, for example, Z - the cone over an anisotropic conic, and R - it's vertex. Then \tilde{Z} has no zero cycles of odd degree, while Z has a rational point.

We will need the following Bertini-type result.

Proposition 7.10 *Let X be smooth quasi-projective variety, and $Z \subset X$ be a proper closed subvariety of it. Then there exists a divisor Y of X which contains Z , and is smooth outside Z , as well as in the generic points of the components of Z .*

7.2 Multiple points excess intersection formula

In this subsection, A^* is any theory in the sense of Definition 2.1. Our main aim here is to prove Proposition 7.21. This analogue of the usual Excess Intersection Formula, where regular embeddings (of smooth varieties) are substituted by strict normal crossing divisors, is a very useful computational tool. To state it, one needs to define the pull-back maps for such divisors. In the case of Algebraic Cobordism, or any theory obtained from it by the change of coefficients, this is just a (small) piece of the theory of refined pull-backs developed by M.Levine-F.Morel (following W.Fulton [4]) (and the Formula is contained in [8, Theorem 6.6.6(2)(a)]). But this piece is sufficient for almost all applications we need. The exception is Subsection 4.3, where the refined pull-backs of more general type appear, and where we have to restrict to *theories of rational type* (= *free theories* of M.Levine-F.Morel) as a result. Despite the fact that our main statements are valid only for *theories of rational type*, I decided to include the treatment of more general case in the places I could. This should be useful in studying analogues of our results for *theories of higher types*. Instead of trying to develop the whole theory of refined pull-backs, I made a short-cut and treated the strict normal crossing divisors only. This bit is much more explicit than the general case. The formulas we get are closely related to *divisor class* of [8], and most of the things we do, in the case of Algebraic Cobordism, should have their analogues in [8, Section 6].

We recall:

Definition 7.11 ([8, Definition 3.1.4]) *Let X be a smooth variety, and $D = \sum_{i=1}^r l_i D_i$ be an effective Weil divisor on X . We call D a divisor with strict normal crossing, if for any $J \subset \{1, \dots, r\}$, the intersection scheme $\cap_{i \in J} D_i$ is a smooth subvariety of X of codimension $= \#(J)$.*

Denote as $|D| \xrightarrow{d} X$ the support $(\cup_{i=1}^r D_i)_{red}$. By $A_*(D)$ we will always mean $A_*(|D|)$. In particular, it does not depend on the multiplicity of the components as long as one is positive. Recall, that we have an exact sequence:

$$0 \leftarrow A_*(D) \leftarrow \oplus_i A_*(D_i) \leftarrow \oplus_{i \neq j} A_*(D_i \cap D_j).$$

Thus, an element of $A_*(D)$ can be thought of as a collection of elements of $A_*(D_i)$ modulo some identifications.

The strict normal crossing divisor has a *divisor class* $[D] \in A^0(D)$ such that $d_*([D]) = c_1^A(\mathcal{O}(D)) \in A^1(X)$. Having $\lambda_i = c_1^A(\mathcal{O}(D_i))$, the idea is to write $[l_1] \cdot_{F_A} \lambda_1 +_{F_A} [l_2] \cdot_{F_A} \lambda_2 +_{F_A} \dots +_{F_A} [l_r] \cdot_{F_A} \lambda_r$ as $\sum_{I \subset \{1, \dots, r\}} (\prod_{i \in I} \lambda_i) \cdot F_I^{l_1, \dots, l_r}(\lambda_1, \dots, \lambda_r)$, where $F_I^{l_1, \dots, l_r}$ is some power series in r variables with A -coefficients, and then define:

Definition 7.12 ([8, Definition 3.1.5])

$$[D] := \sum_{I \subset \{1, \dots, r\}} (\hat{d}_I)_*(1) \cdot F_I^{l_1, \dots, l_r}(\lambda_1, \dots, \lambda_r),$$

where $\hat{d}_I : D_I = \cap_{i \in I} D_i \rightarrow |D|$ is the closed embedding.

The result does not depend on how you "divide" the above formal sum into pieces, but there is some standard way. The convention is (see [8, Subsection 3.1]) to define $F_I^{l_1, \dots, l_r}$ as the sum of those monomials which are made exactly of λ_i , $i \in I$ divided by the $(\prod_{i \in I} \lambda_i)$. For our purposes, though, it will be convenient to be flexible in choosing $F_I^{l_1, \dots, l_r}$, so below it will be any collection of power series satisfying the above equation.

Definition 7.13 Having a divisor $D = \sum_{i=1}^r l_i D_i$ with strict normal crossing on X , we can define the pull-back:

$$d^* : A_*(X) \rightarrow A_{*-1}(D)$$

by the formula

$$d^*(x) = \sum_{I \subset \{1, \dots, r\}} (\hat{d}_I)_* d_I^*(x) \cdot F_I^{l_1, \dots, l_r}(\lambda_1, \dots, \lambda_r),$$

where $d_I : D_I \rightarrow X$ is the regular embedding of the I -th face of D .

Notice, that such a pull-back clearly depends on the multiplicity of the components (in our notations it is manifested only by the target). Also, since for $I \subset J$, for $d_{J/I} : D_J \rightarrow D_I$, we have: $(d_{J/I})_*(1) = \prod_{i \in J \setminus I} \lambda_i$, the projection formula shows that it does not matter, how one chooses the $F_I^{l_1, \dots, l_r}$ (in particular, one can choose these to be zero for $\#(I) > 1$).

Immediately from the definition, we obtain:

Lemma 7.14 The composition $d_* \circ d^* : A^*(X) \rightarrow A^{*+1}(X)$ is the multiplication by $c_1^A(\mathcal{O}(D))$.

□

Let $w : W \rightarrow X \times \mathbb{P}^1$ be a projective map, with W smooth, such that $W_0 = w^{-1}(X \times 0) \xrightarrow{i_0} W$ and $W_1 = w^{-1}(X \times 1) \xrightarrow{i_1} W$ are divisors with strict normal crossing. Let $W_0 \xrightarrow{w_0} X$, $W_1 \xrightarrow{w_1} X$ be natural maps. As a corollary of Lemma 7.14 we get:

Proposition 7.15 In the above situation, $(i_0)_* \circ i_0^* = (i_1)_* \circ i_1^*$ in $A_*(W)$. In particular, $(w_0)_* \circ i_0^* = (w_1)_* \circ i_1^*$

Proof: Observe that $\mathcal{O}_W(W_0) \cong \pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}_W(W_1)$.

□

Let

$$\begin{array}{ccc} E & \xrightarrow{e} & Y \\ \bar{f} \downarrow & & \downarrow f \\ D & \xrightarrow{d} & X. \end{array} \quad (7)$$

be a Cartesian square, where X and Y are smooth and $D \xrightarrow{d} X$ and $E \xrightarrow{e} Y$ are divisors with strict normal crossing (closed codimension 1 subschemes given by principal ideals whose div is a strict normal crossing divisor). Then we can define:

$$\bar{f}^* : A^*(D) \rightarrow A^*(E)$$

as follows. Suppose, $D = \sum_{i=1}^r l_i D_i$, $E = \sum_{j=1}^s m_j E_j$, where D_i and E_j are irreducible components; $\lambda_i = c_1^A(\mathcal{O}(D_i))$, $\mu_j = c_1^A(\mathcal{O}(E_j))$, and $f^*(D_i) = \sum_j p_{i,j} E_j$. If P, L, M are matrices $(p_{i,j})$, (l_i) , (m_j) , then we have: $L \cdot P = M$. Notice, that if $p_{i,j} \neq 0$, for some i and j , then we have the natural map $f_{j,i} : E_j \rightarrow D_i$, and so the map $f_{J,i} : E_J \rightarrow D_i$, for any $J \ni j$. Assume that $F_J^{p_{i,1}, \dots, p_{i,s}} = 0$, if $p_{i,j} = 0$, for some $j \in J$ (notice, that there are no monomials divisible by μ_J in the $\sum_j^{F_A} [p_{i,j}] \cdot_{F_A} \mu_j$, so any "reasonable" choice will do).

Definition 7.16 Let $x = \sum_i (\hat{d}_i)_*(x_i)$, for some $x_i \in A^*(D_i)$. Define:

$$\bar{f}^*(x) := \sum_{i=1}^r \sum_{J \subset \{1, \dots, s\}} (\hat{e}_J)_* f_{J,i}^*(x_i) \cdot F_J^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) \in A^*(E),$$

where we ignore the terms with the zero $F_J^{p_{i,1}, \dots, p_{i,s}}$.

Again, , since for $I \subset J$, for $e_{J/I} : E_J \rightarrow E_I$, we have: $(e_{J/I})_*(1) = \prod_{j \in J \setminus I} \mu_j$, the projection formula shows that it does not matter, how we choose the $F_J^{p_{i,1}, \dots, p_{i,s}}$. Also, it is clear that we get a well-defined map in the case D - smooth irreducible (of multiplicity 1). Below we will show that it is well-defined, in general, but before that we will need to establish several important results.

Lemma 7.17 *There exist $\tilde{F}_J^{m_1, \dots, m_s} \in A[[\mu_1, \dots, \mu_s]]$, such that, for any cartesian diagram as above with D irreducible of multiplicity 1, one has formulas (the second one assuming that f is projective):*

$$\begin{aligned} f^* d_*(x) &= \sum_{J \subset \{1, \dots, s\}} (e_J)_*(f_J)^*(x) \cdot \tilde{F}_J^{m_1, \dots, m_s}(\mu_1, \dots, \mu_s); \\ d^* f_*(y) &= \sum_{J \subset \{1, \dots, s\}} (f_J)_*((e_J)^*(y) \cdot \tilde{F}_J^{m_1, \dots, m_s}(\mu_1, \dots, \mu_s)). \end{aligned}$$

Proof: Consider the triple (W, V, E) , where $V \rightarrow W$ is a codimension 1 regular embedding of smooth varieties, and $E = \sum_{j=1}^s m_j \cdot E_j$ is a divisor with strict normal crossing on V . Suppose, our components are ordered by j . Let $\tilde{W} \xrightarrow{\pi} W$ be the blow up variety Bl_{W, E_1} of W at E_1 , R be its exceptional divisor, and $\tilde{V} \cong V$ be a proper transform of V . Denote as \tilde{E} the divisor $(m_1 - 1) \cdot \tilde{E}_1 + \sum_{j=2}^s m_j \cdot \tilde{E}_j$ on \tilde{V} (with the same ordering of the components). Let $\mathcal{I}_E, \mathcal{I}_{\tilde{E}}$ be the ideal sheaves of E and \tilde{E} in W and \tilde{W} .

The following is a standard result about blow-ups.

Lemma 7.18 *We have:*

- (i) $\pi^*(\mathcal{I}_E) = \mathcal{I}_{\tilde{E}} \cdot \mathcal{I}_R$;
- (ii) $\mathcal{O}_{\tilde{W}}(\tilde{V}) = \pi^* \mathcal{O}_W(V) \otimes \mathcal{O}(1)$, and $\mathcal{O}(1)|_{\tilde{V}} = \mathcal{O}_V(-E_1)$;
- (iii) $R = \mathbb{P}_{E_1}(\mathcal{F})$, where \mathcal{F} is an extension of $\mathcal{O}_W(V)|_{E_1}$ and $\mathcal{O}_V(E_1)$, $\tilde{E}_1 = \mathbb{P}_{E_1}(\mathcal{O}_V(E_1))$, and $R \cap \tilde{E}_i = \tilde{E}_{\{1, i\}} \subset \tilde{E}_1$, for $i > 1$.

□

Let us call the triple $(\tilde{W}, \tilde{V}, \tilde{E})$ - the derivative $(W, V, E)^{(1)}$ of the triple (W, V, E) , and let $(\hat{W}, \hat{V}, 0)$ be the last n -th derivative of (W, V, E) , where $n = (\sum_{j=1}^s m_j)$. Let $n_j = \sum_{i=1}^j m_i$. Let $\mu_j = c_1^A(\mathcal{O}_V(E_j))$, $\eta = c_1^A(\mathcal{O}_W(V))$. It follows from Lemma 7.18 that the components R_k , $k = 1, \dots, n$ of the exceptional divisor R of the map $\hat{W} \rightarrow W$ are as follows: R_k , $n_{j-1} < k \leq n_j$ are isomorphic to $\mathbb{P}_{E_j}(\mathcal{F}_k)$, where the roots of \mathcal{F}_k are expressible in terms of μ_l , $l \leq j$, and η ; R_{n_j} is the last derivative of $\mathbb{P}_{E_j}(\mathcal{F}_{n_j})$ in the triple: $(\mathbb{P}_{E_j}(\mathcal{F}_{n_j}), E_j^{(n_j)}, F_j)$, where $F_j = \sum_{i=j+1}^s m_i \cdot E_{\{i, j\}}^{(n_j)}$. In particular, we have the natural map $\varepsilon_k : R_k \rightarrow E_{j(k)}$, where $n_{j(k)-1} < k \leq n_{j(k)}$. Also, our exceptional divisor R is the one with strict normal crossing with all the components of multiplicity 1.

Consider now the case $W = Y \times \mathbb{P}^1$, $V = Y \times \{0\}$, and E - our divisor with strict normal crossing (so, $\eta = 0$). Let $\mathcal{M}_k = \mathcal{O}(1)_k$ be the linear bundle of the k -th blow up, and $\xi_k = c_1^A(\mathcal{M}_k)$. Then it follows from the formulas of Quillen - Proposition 7.2 and Levine-Morel - Proposition 7.3 that for any $g \in A[[\xi_1, \dots, \xi_n]]$, and any $1 \leq k \leq n$, the element $(\varepsilon_k)_*(g(\xi_1, \dots, \xi_n)) \in A_*(E_{j(k)})$ is expressible in terms of μ_l , $l = 1, \dots, s$ and the coefficients of g . Let $\nu = c_1^A(\mathcal{O}_{\hat{W}}(R))$, and $\nu_k = c_1^A(\mathcal{O}_{\hat{W}}(R_k))$. Since $\nu = \sum_{k=1}^{F_A} \nu_k$, and ν_k , $k = 1, \dots, n$ are expressible in terms of ξ_k , $k = 1, \dots, n$ (and vise-versa), we can write $\nu = \sum_{k=1}^n \nu_k \cdot \theta_k(\xi_1, \dots, \xi_n)$, where $\theta_k \in A(\xi_1, \dots, \xi_n)$ is some power series depending only on m_1, \dots, m_s . Let $\xi = \sum_{k=1}^{F_A} \xi_k$, and $\mu = \sum_{j=1}^{F_A} [m_j] \cdot_{F_A} \mu_j$. Denote $\sum_{n_{j-1} < k \leq n_j} (\varepsilon_k)_*((\xi +_{F_A} \mu) \cdot \theta_k(\xi_1, \dots, \xi_n)) \in A_*(E_j)$ as $\tilde{F}_j^{m_1, \dots, m_s}$. By the above, it can be expressed as the power series in μ_1, \dots, μ_s with coefficients in A , which depends only on m_1, \dots, m_s (but not on the cartesian square). Let us now denote \hat{W} as \hat{Y} .

Let $\tilde{Y} = Bl_{W, E \times \{0\}}$. Then it follows from Lemma 7.18(i), the fact that the double-proj $\text{Proj Proj}(\oplus_a \oplus_b \mathcal{I}^a \mathcal{J}^b)$ does not depend on the order, and the fact that the proj does not change when the ideal is multiplied by a principal one, that $\hat{Y} \rightarrow W$ factors as $\hat{Y} \xrightarrow{\rho} \tilde{Y} \rightarrow W$. And it follows from the same considerations that if $\mathcal{M} = \mathcal{O}(1)$ is the linear bundle of the blow up $\tilde{Y} \rightarrow W$, then $\rho^*(\mathcal{M}) = \otimes_k \mathcal{M}_k$ (notice, that \tilde{Y} is usually non-smooth).

Consider the diagram with the left squares transversal cartesian:

$$\begin{array}{ccccc} D & \xrightarrow{d} & X & \xleftarrow{f} & Y \\ i_D \downarrow & & \downarrow i_X & & \downarrow i_Y \\ \hat{D} & \xrightarrow{\hat{d}} & \hat{X} & \xleftarrow{\hat{f}} & \hat{Y} \\ l_D \uparrow & & i_T \uparrow & & \uparrow i_{R_k} \\ D & \xrightarrow{\tilde{d}} & T & \xleftarrow{\tilde{f}_k} & R_k, \end{array}$$

where $\hat{X} = Bl_{X \times \mathbb{P}^1, D \times \{0\}}$; $T = \mathbb{P}_D(\mathcal{O}(D) \oplus \mathcal{O}) \xrightarrow{\varepsilon} D$ be the exceptional divisor with the section $\mathbb{P}_D(\mathcal{O}) \xrightarrow{\tilde{d}} T$; \hat{f} be the natural composition $\hat{Y} \xrightarrow{\rho} \tilde{Y} \rightarrow \hat{X}$; $\hat{D} = Bl_{D \times \mathbb{P}^1, D \times \{0\}} = D \times \mathbb{P}^1$; and $\hat{D} \xrightarrow{\hat{d}} \hat{X}$, $D \times \{1\} \xrightarrow{i_D} \hat{D}$, $X \times \{1\} \xrightarrow{i_X} \hat{X}$, and $Y \times \{1\} \xrightarrow{i_Y} \hat{Y}$ be the natural embeddings. Let $\hat{X} \xrightarrow{\pi_X} X$, $\hat{Y} \xrightarrow{\pi_Y} Y$ and $\hat{D} \xrightarrow{\pi_D} D$ be the natural projections. Let \mathcal{L} be the line bundle $\mathcal{O}(1)$ of the blow up $\hat{X} \rightarrow X \times \mathbb{P}^1$.

Since the proper preimage X_0 of $X \times \{0\}$ in \hat{X} does not meet \hat{D} , and \hat{f} maps the proper preimage Y_0 of $Y \times \{0\}$ (in \hat{Y}) to X_0 , one has

$$c_1^A(\mathcal{O}(Y_0)) \cdot \hat{f}^* \hat{d}_*(anything) = 0, \quad \text{and} \quad \hat{d}^* \hat{f}_*(c_1^A(\mathcal{O}(Y_0)) \cdot anything) = 0.$$

We also have: $c_1^A(\mathcal{O}(1)) = c_1^A(\mathcal{O}(Y_0)) +_{F_A} c_1^A(R)$, $\hat{f}^*(\mathcal{L}) = \rho^*(\mathcal{M}) = \otimes_k \mathcal{M}_k$, and $f^*(\lambda) = \mu$. Combining this with the equalities:

$$\tilde{d}_*(x) = c_1^A(\mathcal{O}(D) \otimes \mathcal{O}(1)) \cdot \varepsilon^*(x), \quad \tilde{d}^*(u) = \varepsilon_*(c_1^A(\mathcal{O}(D) \otimes \mathcal{O}(1)) \cdot u),$$

the fact that the left squares are transversal cartesian and the projection formula, we get:

$$\begin{aligned} f^* d_*(x) &= f^* i_X^* \hat{d}_* \pi_D^*(x) = i_Y^* \hat{f}^* \hat{d}_* \pi_D^*(x) = (\pi_Y)_*(c_1^A(\mathcal{O}(1)) \cdot \hat{f}^* \hat{d}_* \pi_D^*(x)) = (\pi_Y)_*(\nu \cdot \hat{f}^* \hat{d}_* \pi_D^*(x)) = \\ &= \sum_{k=1}^n (\pi_Y)_*(\nu_k \cdot \theta_k \cdot \hat{f}^* \hat{d}_* \pi_D^*(x)) = \sum_{k=1}^n (\pi_Y)_*(i_{R_k})_*(i_{R_k})^*(\theta_k \cdot \hat{f}^* \hat{d}_* \pi_D^*(x)) = \sum_{k=1}^n (e_{j(k)})_*(\varepsilon_k)_*(\theta_k \cdot \tilde{f}_k^* \tilde{d}_*(x)) \\ &= \sum_{k=1}^n (e_{j(k)})_*(f_{j(k)}^*(x) \cdot (\varepsilon_k)_*(\theta_k \cdot (\xi +_{F_A} \mu))) = \sum_{j=1}^s (e_j)_*(f_j^*(x) \cdot \tilde{F}_j^{m_1, \dots, m_s}), \end{aligned}$$

which settles the first equality. In a similar way, we get:

$$\begin{aligned} d^* f_*(y) &= (\pi_D)_* \hat{d}^* \hat{f}_*(i_Y)_*(y) = (\pi_D)_* \hat{d}^* \hat{f}_*(c_1^A(\mathcal{O}(1)) \cdot \pi_Y^*(y)) = (\pi_D)_* \hat{d}^* \hat{f}_*(\nu \cdot \pi_Y^*(y)) = \\ &= \sum_{k=1}^n (\pi_D)_* \hat{d}^* \hat{f}_*(\nu_k \cdot \theta_k \cdot \pi_Y^*(y)) = \sum_{k=1}^n (\pi_D)_* \hat{d}^* \hat{f}_*(i_{R_k})_*(i_{R_k})^*(\theta_k \cdot \pi_Y^*(y)) = \sum_{k=1}^n \tilde{d}^*(\tilde{f}_k)_*(\theta_k \cdot \varepsilon_k^*(e_{j(k)})^*(y)) \\ &= \sum_{k=1}^n (f_{j(k)})_*(e_{j(k)}^*(y) \cdot (\varepsilon_k)_*((\xi +_{F_A} \mu) \cdot \theta_k)) = \sum_{j=1}^s (f_j)_*(e_j^*(y) \cdot \tilde{F}_j^{m_1, \dots, m_s}), \end{aligned}$$

which settles the second. □

Lemma 7.19 *Suppose, D is irreducible of multiplicity 1. Then*

(1)

$$e_* \circ \bar{f}^* = f^* \circ d_*.$$

(2) *Suppose, f is projective. Then*

$$\bar{f}_* \circ e^* = d^* \circ f_*.$$

Proof: Let $X = \mathbb{P}^\infty$, $Y = (\mathbb{P}^\infty)^{\times s}$, and $f : Y \rightarrow X$ is the composition of the Segre and Veronese embeddings:

$$\text{Seg}^s(\text{Ver}(m_1), \dots, \text{Ver}(m_s)) : (\mathbb{P}^\infty)^{\times s} \rightarrow \mathbb{P}^\infty.$$

Let y_i be some linear function (coordinate) on $(\mathbb{P}^\infty)_i$, and $x = y_1^{m_1} \cdot \dots \cdot y_s^{m_s}$ be the respective linear function on X . Let $D \rightarrow X$ be the divisor defined by the equation $x = 0$, and E_i be the divisor on Y defined by the equation $y_i = 0$. Clearly, these are smooth irreducible, and $f^*(D) = \sum_{i=1}^s m_i \cdot E_i$ is a divisor with strict normal crossing. Thus, by Lemma 7.17,

$$f^* d_*(1) = \sum_{J \subset \{1, \dots, s\}} \mu_J \cdot \tilde{F}_J^{m_1, \dots, m_s}(\mu_1, \dots, \mu_s),$$

for some power series $\tilde{F}_J^{m_1, \dots, m_s} \in A[[\mu_1, \dots, \mu_s]]$. But the left hand side is $f^*(\lambda) = \sum_j^{F_A} [m_j] \cdot_{F_A} \mu_j$, and our variables μ_1, \dots, μ_s are independent this time: $A^*(Y) = A[[\mu_1, \dots, \mu_s]]$. Thus, the power series $\tilde{F}_J^{m_1, \dots, m_s}$ satisfy the equation defining $F_J^{m_1, \dots, m_s}$, and hence,

$$\begin{aligned} f^* d_*(x) &= \sum_{J \subset \{1, \dots, s\}} (e_J)_*(f_J)^*(x) \cdot \tilde{F}_J^{m_1, \dots, m_s}(\mu_1, \dots, \mu_s) = e_* \bar{f}^*(x); \\ d^* f_*(y) &= \sum_{J \subset \{1, \dots, s\}} (f_J)_*((e_J)^*(y) \cdot \tilde{F}_J^{m_1, \dots, m_s}(\mu_1, \dots, \mu_s)) = \bar{f}_* e^*(y). \end{aligned}$$

□

Now we can prove that \bar{f}^* is well-defined.

Lemma 7.20 *The element $\bar{f}^*(x)$ does not depend on the presentation $x = \sum_i (\hat{d}_i)_*(x_i)$, and gives a well-defined map: $\bar{f}^* : A^*(D) \rightarrow A^*(E)$.*

Proof: Let, $\{i, k\} \subset \{1, \dots, r\}$. We can choose $F_J^{p_{i,1}, \dots, p_{i,s}}$ and $F_J^{p_{k,1}, \dots, p_{k,s}}$ to be zero, for all $\#(J) > 1$. Let $y \in A^{*-1}(D_{\{i,k\}})$.

Let us (temporarily) use the notation $(e_{\{j,j\}/j})_* : A^{*-1}(E_j) \rightarrow A^*(E_j)$ for the multiplication by μ_j map, and $f_{\{j,j\},\{i,k\}}$ for the map $f_{j,\{i,k\}}$.

Suppose j is such that $p_{k,j} = 0$. Then the divisor $f^*(D_k) = \sum_l p_{k,l} \cdot E_l$ is transversal to E_j . Thus, in the cartesian square:

$$\begin{array}{ccc} F & \longrightarrow & E_j \\ \downarrow & & \downarrow f_{j,i} \\ D_{\{i,k\}} & \longrightarrow & D_i, \end{array}$$

the divisor F is $\sum_l p_{k,l} E_{\{j,l\}}$. Then, by Lemma 7.19,

$$f_{j,i}^*(d_{\{i,k\}/i})_*(y) = \sum_l (e_{\{j,l\}/j})_* f_{\{j,l\},\{i,k\}}^*(y) \cdot F_l^{p_{k,1}, \dots, p_{k,s}}(\mu_1, \dots, \mu_s).$$

Suppose j is such that $p_{k,j} \neq 0$. Then, either $p_{i,j} = 0$, and so $F_j^{p_{i,1}, \dots, p_{i,s}} = 0$, or the map $f_{j,i} : E_j \rightarrow D_i$ factors through $f_{j,\{i,k\}} : E_j \rightarrow D_{\{i,k\}}$. In any case, we get:

$$f_{j,i}^*(d_{\{i,k\}/i})^*(y) \cdot F_j^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) = f^*(\lambda_k) \cdot f_{j,\{i,k\}}^*(y) \cdot F_j^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) = \sum_l (e_{\{j,l\}/j})^* f_{\{j,l\},\{i,k\}}^*(y) \cdot F_l^{p_{k,1}, \dots, p_{k,s}}(\mu_1, \dots, \mu_s) \cdot F_j^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s).$$

Combining both cases, we get:

$$\begin{aligned} \sum_j (\hat{e}_j)^* f_{j,i}^*(d_{\{i,k\}/i})^*(y) \cdot F_j^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) &= \\ \sum_{l,j} (\hat{e}_j)^* (e_{\{j,l\}/j})^* f_{\{j,l\},\{i,k\}}^*(y) \cdot F_l^{p_{k,1}, \dots, p_{k,s}}(\mu_1, \dots, \mu_s) \cdot F_j^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) &= \\ \sum_{l,j} (\hat{e}_l)^* (e_{\{j,l\}/l})^* f_{\{j,l\},\{i,k\}}^*(y) \cdot F_l^{p_{k,1}, \dots, p_{k,s}}(\mu_1, \dots, \mu_s) \cdot F_j^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) &= \\ \sum_l (\hat{e}_l)^* f_{l,k}^*(d_{\{i,k\}/k})^*(y) \cdot F_j^{p_{k,1}, \dots, p_{k,s}}(\mu_1, \dots, \mu_s). \end{aligned}$$

This shows that \bar{f}^* is a well-defined map on $A^*(D) = (\oplus A^*(D_i))/\text{image}(\oplus_{i,k} A^{*-1}(D_{\{i,k\}}))$. \square

Proposition 7.21 (Multiple points excess intersection formula)

In the above situation, we have:

(1)

$$e_* \circ \bar{f}^* = f^* \circ d_*.$$

(2) *Suppose, f is projective. Then*

$$\bar{f}_* \circ e^* = d^* \circ f_*.$$

Proof: (1) Let D_i be an irreducible component of D . Then in the cartesian square:

$$\begin{array}{ccc} F & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ D_i & \xrightarrow{d_i} & X \end{array}$$

we have: $F = \sum_j p_{i,j} \cdot E_j$. Then the statement follows from Lemma 7.19.

(2) We can assume that $F_I^{l_1, \dots, l_r} = 0$, for $\#(I) > 1$. By Lemma 7.19, we have:

$$\begin{aligned} d^* f_*(y) &= \sum_{i=1}^r (\hat{d}_i)^* d_i^* f_*(y) \cdot F_i^{l_1, \dots, l_r}(\lambda_1, \dots, \lambda_r) = \\ \sum_{i=1}^r (\hat{d}_i)^* \left(\sum_{J \subset \{1, \dots, s\}} (\bar{f}_{J,i})^* e_J^*(y) \cdot F_J^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) \right) \cdot F_i^{l_1, \dots, l_r}(\lambda_1, \dots, \lambda_r) &= \\ \bar{f}_* \left(\sum_{i=1}^r \sum_{J \subset \{1, \dots, s\}} (\hat{e}_J)^* e_J^*(y) \cdot F_J^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) \cdot f^* F_i^{l_1, \dots, l_r}(\lambda_1, \dots, \lambda_r) \right) &= \\ \bar{f}_* \left(\sum_{J \subset \{1, \dots, s\}} (\hat{e}_J)^* e_J^*(y) \cdot F_J^{m_1, \dots, m_s}(\mu_1, \dots, \mu_s) \right) &= \bar{f}_* e^*(y). \end{aligned}$$

□

We will need some results concerning functoriality of \overline{u}^* . Suppose,

$$\begin{array}{ccccc} D & \xleftarrow{\overline{u}} & E & \xleftarrow{\overline{v}} & F \\ d \downarrow & & \downarrow e & & \downarrow f \\ X & \xleftarrow{u} & Y & \xleftarrow{v} & Z. \end{array}$$

be the cartesian diagram, where X, Y, Z are smooth, and D, E and F are divisors with strict normal crossing.

Proposition 7.22 *In the above situation,*

$$(\overline{u} \circ \overline{v})^* = \overline{v}^* \circ \overline{u}^*.$$

Proof: Let $D = \sum_{i=1}^r l_i \cdot D_i$, $E = \sum_{j=1}^s m_j \cdot E_j$, and $F = \sum_{k=1}^t n_k \cdot F_k$. Let $u^*(D_i) = \sum_{j=1}^s p_{i,j} \cdot E_j$, and $v^*(E_j) = \sum_{k=1}^t q_{j,k} \cdot F_k$. If L, M, N, P, Q are the respective matrices, then $L \cdot P = M$ and $M \cdot Q = N$. The matrix of $(u \circ v)^*$ is then given by $R = P \cdot Q$. Let $\lambda_i = c_1^A(\mathcal{O}(D_i))$, $\mu_j = c_1^A(\mathcal{O}(E_j))$, $\nu_k = c_1^A(\mathcal{O}(F_k))$. Now, we can assume that $F_j^{p_{i,1}, \dots, p_{i,s}} = 0$, for $\#(J) > 1$. Then, for $x = \sum_i (\hat{d}_i)_*(x_i)$, we have:

$$\begin{aligned} \overline{v}^* \overline{u}^*(x) &= \overline{v}^* \sum_{i=1}^r \sum_{j=1}^s (\hat{e}_j)_* u_{j,i}^*(x_i) \cdot F_j^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s) = \\ &= \sum_{i=1}^r \sum_{j=1}^s \sum_{K \subset \{1, \dots, t\}} v_{K,j}^* u_{j,i}^*(x_i) \cdot v_{K,j}^*(F_j^{p_{i,1}, \dots, p_{i,s}}(\mu_1, \dots, \mu_s)) \cdot F_K^{q_{j,1}, \dots, q_{j,t}}(\nu_1, \dots, \nu_t) = \\ &= \sum_{i=1}^r \sum_{K \subset \{1, \dots, t\}} (v \circ u)_{K,i}^*(x_i) \cdot F_K^{r_{i,1}, \dots, r_{i,t}}(\nu_1, \dots, \nu_t) = (\overline{u} \circ \overline{v})^*(x). \end{aligned}$$

□

Finally, in the case of a *free theory* in the sense of Levine-Morel - see [8, Remark 2.4.14] (by Proposition 4.8, these theories are exactly our *theories of rational type*), e^* appears to be the same as a *refined pull-back* morphism.

Lemma 7.23 *Let $A^* = \Omega^* \otimes_{\mathbb{L}} A$ be a theory obtained from Algebraic Cobordism by change of coefficients. Then for any square (7), we have:*

$$e^* = d^! \quad \text{and} \quad \overline{f}^* = f^!.$$

Proof: The first identity follows from Lemma 6.6.2, Lemma 6.5.6, Definition 6.5.1, and definitions of Subsection 6.2.1 of [8]. The second identity needs to be checked only for the case where D is a smooth divisor and f is a regular embedding, where, in the case of codimension 1, it follows from the first identity, and the general case follows from the deformation to the normal cone construction. □

In this light, our Multiple Points Excess Intersection Formula (Proposition 7.21) and Proposition 7.22 are just particular cases of [8, Theorem 6.6.6(2)(a),(3)].

8 Resolution of singularities

In this section we list the results related to Resolution of Singularities and the Weak Factorization Theorem which are widely used throughout the text.

Definition 8.1 *Let X be a smooth variety and D - a divisor with strict normal crossing on it. By a permitted blow-up w.r.to D we will understand such a sequence of blow-ups with smooth centers $R_i \subset X_i$:*

$$\tilde{X} = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X$$

such that, for the exceptional divisor E_i of $\pi^i = \pi_1 \circ \dots \circ \pi_i : X_i \rightarrow X$, and the total transform $(\pi^i)^(D)$, the divisor $E_i + (\pi^i)^*(D)$ has strict normal crossing, and R_i has normal crossing with it.*

If D is empty, we will call it just a permitted blow-up.

Theorem 8.2 (Hironaka, [5]) *Let Z be a subvariety of a smooth variety X . Then there exists a permitted blow-up $\pi : \tilde{X} \rightarrow X$ such that:*

- (1) *All the centers R_i are lying over the singular locus of Z .*
- (2) *The strict transform $\tilde{Z} \subset \tilde{X}$ of Z is smooth and has normal crossing with E_n .*

Theorem 8.3 (Hironaka, [5]) *Let $f : X \dashrightarrow Y$ be a rational map of reduced varieties. Then there is a permitted blow-up $\pi : \tilde{X} \rightarrow X$ such that:*

- (1) *All the centers R_i are lying over the locus of X where it is not smooth, or f is not a morphism.*
- (2) *The rational map $f \circ \pi : X \rightarrow Y$ is a morphism.*

Theorem 8.4 (Hironaka, [5], see also [1, 1.2.3] and [2]) *Let \mathcal{I} be a sheaf of ideals on a smooth variety X , and $U \subset X$ be an open subvariety such that $\mathcal{I}|_U$ is an ideal sheaf of a divisor with strict normal crossing. Then there is a permitted blow-up $\pi : \tilde{X} \rightarrow X$ with centers outside U such that the total transform $\pi^*(\mathcal{I})$ is an ideal of a strict normal crossing divisor \tilde{E} .*

There is also a relative to divisor D version (see [1, 1.2.2] and [2]).

Proposition 8.5 *Let X be smooth quasi-projective variety, $Z \subset X$ - a closed subvariety, and D - a divisor with strict normal crossing on X . Then there exists a permitted w.r.to D blow up $\tilde{X} \xrightarrow{\pi} X$ with centers over Z such that $\pi^{-1}(Z) \cup \pi^{-1}(D)$ is a divisor with strict normal crossing.*

The following result is the Weak Factorization Theorem - [1, Theorem 0.3.1], see also [25].

Theorem 8.6 (Abramovich-Karu-Matsuki-Wlodarczyk) *Let $\theta : X_1 \dashrightarrow X_2$ be birational map of smooth proper varieties over k , which is an isomorphism on the open set $U \subset X_1$. Then θ can be factored into a sequence of blowings up and blowings down with nonsingular centers disjoint from U . Namely, to any such θ we can associate a diagram:*

$$X_1 = Y_0 \xrightarrow{\varphi_1} Y_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{i-1}} Y_{i-1} \xrightarrow{\varphi_i} Y_i \xrightarrow{\varphi_{i+1}} \dots \xrightarrow{\varphi_{l-1}} Y_{l-1} \xrightarrow{\varphi_l} Y_l = X_2$$

where

- (1) $\theta = \varphi_l \circ \varphi_{l-1} \circ \dots \circ \varphi_2 \circ \varphi_1$,
- (2) φ_i are isomorphisms on U , and

- (3) either φ_i , or φ_i^{-1} is a blow up morphism with smooth center disjoint from U .
- (4) *Functoriality*: if $g : \theta \rightarrow \theta'$ is an absolute isomorphism carrying U to U' , and $\varphi'_i : Y'_{i-1} \dashrightarrow Y_i$ is the factorization of φ' , then the resulting rational maps $g_i : Y_i \dashrightarrow Y'_i$ is an absolute isomorphism.
- (5) There is an index i_0 such that, for $i \leq i_0$, the map $Y_i \dashrightarrow X_1$ is projective map, while for $i \geq i_0$, $Y_i \dashrightarrow X_2$ is projective map.
- (6) Let $E_i \subset Y_i$ be the exceptional divisor of $Y_i \rightarrow X_1$ (respectively, of $Y_i \rightarrow X_2$) in case $i \leq i_0$ (respectively, $i \geq i_0$). Then the above centers of blow up have normal crossing with E_i . If, moreover, $X_1 \setminus U$ (respectively, $X_2 \setminus U$) is a normal crossing divisor, then the centers of blow up have normal crossing with the inverse images of this divisor.

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